25-th Mathematical Olympiad 1988

December 1988

Time allowed: 3.5 hours

1. Find all integers a, b, and c for which

$$(x-a)(x-10)+1=(x+b)(x+c) \quad \text{for all} x$$

2. Points P and Q lie on the sides AB and AC respectively of triangle ABC and are distinct from A. The lengths AP and AQ are denoted by x and y repectively, with the convention that $x>0$ if P is on the same side of A as B, and $x<0$ on the opposite side; and similarly for y. Show that PQ passes through the centroid of the triangle if and only if

$$3xy=bx+cy,$$

where $b=AC$ and $c=AB$.

3. The lines OA, OB, and OC are mutually perpendicular. Express the area of triangle ABC in terms of the areas of triangles OBC, OCA, and OAB.

4. Consider following the triangle of numbers. Each number is the sum of three numbers in the previous row: the number above it and the numbers immediately to the left and right of that number. If there is no number in one or more of these positions, 0 is used. Prove that, from the third row on, every row contains at least one even number.

$$\begin{array}{ccccccccc}
 & & & & 1 & & & & \\
 & & & 1 & 1 & 1 & & & \\
 & & 1 & 2 & 3 & 2 & 1 & & \\
 & 1 & 3 & 6 & 7 & 6 & 3 & 1 & \\
. & . & . & . & . & . & . & . & .
\end{array}$$

5. None of the angles of a triangle ABC exceeds 90°. Prove that

$$\sin A+\sin B+\sin C>2.$$

6. The sequence $\{a_n\}$ of integers is defined by $a_1=2$, $a_2=7$, and

$$-\frac{1}{2}<a_{n+1}-\frac{a_n^2}{a_{n-1}} \quad , \quad \text{for } n\geq 2$$

Prove that a_n is odd for all $n>1$.