26-th Mathematical Olympiad 1990

January 1990

Time allowed: 3.5 hours

1. Find a positive integer the first digit of which is 1 and which has the property that, if this digit is transferred to the end of the number, the number is tripled.

2. $ABCD$ is a square and P is a point on the line AB. Find the maximum and minimum values of the ratio PC/PD, showing that these occur for the points P satisfying $AP \cdot BP = AB^2$.

3. The angles A, B, C, and D of a convex quadrilateral satisfy the relation

$$\cos A + \cos B + \cos C + \cos D = 0.$$

Prove that $ABCD$ is either a trapezium or is cyclic.

4. A coin is biased so that the probability of obtaining a head is p, $0 < p < 1$. Two players, A and B, throw the coin in turn until one of the sequences HHH or HTH occurs. If the sequence HHH occurs first, then A wins. If HTH occurs first, then B wins. For what value of p is the game fair (that , is, such that A and B have an equal chance of winning)?

5. The diagonals of a convex quadrilateral $ABCD$ intersect at O. The centroids of triangles AOD and BOC are P and Q; the orthocentres of triangles AOB and COD are R and S. Prove that PQ is perpendicular to RS.

[*Note*: The centroid of a triangle is the intersection of the lines joining each vertex to the midpoint of the opposite side; the onhocentre is the intersection of the altitudes.]

6. Prove that if x and y are rational numbers satisfying the equation

$$x^5 + y^5 = 2x^2y^2$$

then $1 - xy$ is the square of a rational number.