27-th British Mathematical Olympiad 1991

January 16, 1991

Time: 3.5 hours.

1. Prove that $3^n + 2 \cdot 17^n$ is not a perfect square for any nonnegative integer n.

2. Find all positive integers k such that the polynomial $x^{2k+1} + x + 1$ is divisible by $x^k + x + 1$. For each such k, specify those n for which $x^n + x + 1$ is divisible by $x^k + x + 1$.

3. A quadrilatereal $ABCD$ is inscribed in a circle of radius r. The diagonals AC and BD meet at E. Prove that if $AC \perp BD$, then $EA^2 + EB^2 + EC^2 + ED^2 = 4r^2$. Is the converse necessarily true?

4. If x, y, z are positive real numbers satisfying the equality $xyz(x+y+z) = 1$, find the minimum value of $(x + y)(y + z)$.

5. Find the number of permutations $p_1, p_2, \dots, p_6$ of the numbers $1, 2, \dots, 6$ with the property: For no n with $1 \le n \le 5$ do $p_1, \dots, p_n$ form a permutation of $1, \dots, n$.

6. Show that if x and y are natural numbers such that $x^2 + y^2 - x$ is divisible by $2xy$, then x is a perfect square.

7. A ladder of length l rests against a vertical wall. Suppose there is a rung on the ladder which has the same distance d from both the wall and the (horizontal) ground. Find explicitly, in terms of l and d, the height h from the ground that the ladder reaches up the wall.

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