19-th British Mathematical Olympiad 1983

March 10, 1983

Time: 3.5 hours.

1. In a triangle ABC with $AB = AC$, O is the circumcenter, D the midpoint of AB, and E the centroid of $\triangle ACD$. Prove that OE is perpendicular to CD.

2. The Fibonacci sequence (f_n) is defined by $f_1 = f_2 = 1$ and $f_{n+1} = f_n + f_{n-1}$ for $n \geq 2$. Prove that there are unique integers a, b, m with $0 < a, b < m$ such that m divides $f_n - anb^n$ for all $n > 0$.

3. The sequence (x_n) is defined by $x_1 = a \neq -1$ and $x_{n+1} = x_n^2 + x_n$ for $n \geq 1$. Let $y_n = \dfrac{1}{1 + x_n}$, and let S_n be the sum and P_n the product of $y_1, y_2, \dots, y_n$ for $n \in \mathbb{N}$. Prove that $aS_n + P_n = 1$ for all n.

4. The two cylindrical surfaces $x^2 + z^2 = a^2$ ($z > 0$, $|y| \leq a$) and $y^2 + z^2 = a^2$ ($z > 0$, $|x| \leq a$) intersect and with the plane $z = 0$ enclose a dome-like shape which is here called a *cupola*. The cupola is placed on top of a vertical tower of height h whose horizontal cross section is a square of side $2a$. Find the shortest distance from the highest point of the cupola to a corner of the base of the tower, over the surface of the cupola and tower.

5. Given 10 points inside a circle of diameter 5 inches, prove that some two are on a distance less than 2 inches.

6. Consider the equation in x

$$\sqrt{2p + 1 - x^2} + \sqrt{3x + p + 4} = \sqrt{x^2 + 9x + 3p + 9}$$

in which x and p are real numbers. Show that the equation implies that $(x^2 + x - p)(x^2 + 8x + 2p + 9) = 0$ and hence find all real p for which the equation has exactly one real solution.