23-th Mathematical Olympiad 1987

January 1987

Time allowed: 3.5 hours

1. (a) Find, with proof, all integer solutions of $a^3 + b^3 = 9$.

   (b) Find, with proof, all integer solutions of

$$35x^3 + 66x^2y + 42xy^2 + 9y^3 = 9$$

.

2. In a triangle ABC, $\angle BAC = 100^\circ$ and $AB = AC$. A point D is chosen on the side AC so that $\angle ABD = \angle CBD$. Prove that $AD + DB = BC$.

3. Find, with proof, the value of the limit as $n \to \infty$ of the quotient

$$\frac{\sum_{r=0}^{n} \binom{2n}{2r} \cdot 2^r}{\sum_{r=0}^{n-1} \binom{2n}{2r+1} \cdot 2^r}$$

4. Let $P(x)$ be any polynomial with integer coefficients such that

$$P(21) = 17 \quad , \quad P(32) = -247 \quad , \quad P(37) = 33$$

   Prove that if $P(N) = N + 51$, for some integer N, then $N = 26$.

5. A line parallel to the side BC of an acute angled triangle ABC cuts the side AB at F and the side AC at E. Prove that the circles on BE and CF as diameters intersect on the altitude of the triangle drawn from A perpendicular to BC.

6. If x, y, and z are positive real numbers, find, with proof, the maximum value of the expression

$$\frac{xyz}{(1+x)(x+y)(y+z)(z+16)}$$

7. Let n and k be arbitrary positive integers. Prove that there exists a positive integer x such that $\frac{1}{2}x(x+1) - k$ is divisible by 2^n.