24-th Mathematical Olympiad 1987

December 1987

Time allowed: 3.5 hours

1. Find all real solutions x of the equation

$$\sqrt{x+1972098-1986\sqrt{x+986049}}+\sqrt{x+1974085-1988\sqrt{x+986049}}=1$$

[*Note*: $\sqrt{y}$ always denotes the non-negative square root of y.]

2. Find all real-valued functions f which are defined on the set D of natural numbers $x \geq 10$, and which satisfy the functional equation

$$f(x+y)=f(x)\cdot f(y)$$

for all $x, y \in D$.

3. Find a pair of integers r, s such that $0 < s < 200$ and

$$\frac{45}{61} > \frac{r}{s} > \frac{59}{80}$$

Prove that there is exactly one such pair r, s.

4. The triangle ABC has orthocentre H. The feet of the perpendiculars from H to the internal and external bisectors of angle BAC (which is not a right angle) are P and Q. Prove that PQ passes through the midpoint of BC.

5. For any two integers m and n with $0 \leq m \leq n$, numbers $d(n,m)$ are defined by

$$d(n,0)=An,n)=1 \quad \text{for all } n \geq 0$$

and

$$md(n,m)=md(n-1,m)+(2n-m)d(n-1,m-1) \quad \text{for } 0 < m < n.$$

Prove that all of the $d(n,m)$ are integers.

6. Show that, if x and y are real numbers such that $7x^2+3xy+3y^2=1$, then the least positive value of $(x^2+y^2)/y$ is $\frac{1}{2}$.