20-th British Mathematical Olympiad 1984

March 13, 1984

Time: 3.5 hours.

1. Let P, Q and R be arbitrary points on the sides BC, CA and AB of triangle ABC, respectively. Prove that the triangle with the vertices at the circumcenters of the triangles AQR, BRP, CPQ is similar to triangle ABC.

2. For every positive integer n, let a_n be the number of those integers r with $0 \le r \le n$ for which $\binom{n}{r}$ leaves remainder 1 on division by 3, and let b_n be the number of those for which the remainder is 2. Prove that $a_n > b_n$ for all n.

3. (a) Prove that for all integers m,
$$\left(2-\frac{1}{m}\right)\left(2-\frac{3}{m}\right)\left(2-\frac{5}{m}\right)\cdots\left(2-\frac{2m-1}{m}\right) \le \frac{2^m m!}{m^m}.$$
   (b) Prove that if a, b, c, d, e are positive real numbers, then
$$\left(\frac{a}{b}\right)^4+\left(\frac{b}{c}\right)^4+\left(\frac{c}{d}\right)^4+\left(\frac{d}{e}\right)^4+\left(\frac{e}{a}\right)^4 \ge \frac{b}{a}+\frac{c}{b}+\frac{d}{c}+\frac{e}{d}+\frac{a}{e}.$$

4. For each positive integer N, find the number of solutions x of the equation
$$x^2-[x^2]=(x-[x])^2$$
lying in the interval $[1, N]$.

5. A plane cuts a right circular cone with vertex V in an ellipse E and meets the axis of the cone at C. Let A be an extremity of the major axis of E. Prove that the area of the curved surface of the slant cone with vertex V and base E equals $\frac{VA}{AC}$ times the area of E.

6. Suppose a and m are natural numbers and x an integer such that m divides a^2x-a. Prove that there is an integer y such that m divides both a^2y-a and ay^2-y.

7. A quadrilateral $ABCD$ has an inscribed circle. Define
$$u_{AB}=p_1\sin\angle DAB+p_2\sin\angle ABC,$$
where p_1 and p_2 are the perpendiculars from A and B, respectively, to side CD. Quantities u_{BC}, u_{CD}, and u_{DA} are analogously defined. Show that $u_{AB}=u_{BC}=u_{CD}=u_{DA}$.

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