21-th Mathematical Olympiad 1985

January 1985

Time allowed: 3.5 hours

1. Circles S_1 and S_2 both touch a straight line p at the point P. All points of S_2, except P, are in the interior of S_1. The straight line q: (i) is perpendicular to p; (ii) touches S_2 at R; (iii) cuts p at L; and (iv) cuts S_1 at N and M, where M is between L and R.

   (a) Prove that RP bisects $\angle MPN$.

   (b) If MP bisects $\angle RPL$, find, with proof, the ratio of the areas of the circles S_1, and S_2.

2. Given any three numbers a, b, and c between 0 and 1, prove that not all of the expressions $a(1-b)$, $b(1-c)$, and $c(1-a)$ can be greater than $\frac{1}{4}$.

3. Given any two non-negative integers n and m with $n \geq m$, prove that

$$\binom{n}{m} + 2\binom{n-1}{m} + 3\binom{n-2}{m} + \cdots + (n+1-m)\binom{m}{m} = \binom{n+2}{m+2}$$

   Note: $\binom{r}{s}$ denotes the binomial coefficient

$$\binom{r}{s} = \frac{r(r-1)(r-2)\cdots(r-s+1)}{s!}$$

4. The sequence f_n is defined by $f_0 = 1$ and $f_1 = c$, where c is a positive integer, and for all $n \geq 1$ by

$$f_n = 2f_{n-1} - f_{n-2} + 2.$$

   Prove that, for each $k \geq 0$, there exists h such that $f_k f_{k+1} = f_h$.

5. A circular hoop of radius 4 cm is held fixed in a horizontal plane. A cylinder with radius 4 cm and length 6 cm rests on the hoop with its axis horizontal, and with each of its two circular ends touching the hoop at two points. The cylinder is free to move subject to the condition that each of its circular ends always touches the hoop at two points. Find, with proof, the locus of the centre of one of the cylinder's circular ends.

6. Show that the equation $x^2 + y^2 = z^5 + z$ has infinitely many solutions in positive integers x, y, and z having no common factor greater than 1.