22-th Mathematical Olympiad 1986

January 1986

Time allowed: 3.5 hours

1. Reduce the fraction $\frac{N}{D}$ to its lowest terms when

$$N = 2244851485148514627 \quad , \quad D = 8118811881188118000$$

2. A circle S of radius R has two parallel tangents, t_1 and t_2. A circle S_1 of radius r_1 touches S and t_1; a circle S_2 of radius r_2 touches S and t_2; S_1 also touches S_2. All of the contacts are external. Calculate R in terms of r_1 and r_2.

3. Prove that if m, n, and r are positive integers and

$$1 + m + n\sqrt{3} = (2 + \sqrt{3})^{2r-1}$$

then m is a perfect square.

4. Find, with proof, the largest real number K (independent of a, b, and c) such that the inequality

$$a^2 + b^2 + c^2 > K(a + b + c)^2$$

holds for the lengths a, b, and c of the sides of any *obtuse-angled* triangle.

5. Find, with proof, the number of *permutations*

$$a_1, a_2, a_3, \ldots, a_n \quad \text{of} \quad 1, 2, 3, \ldots, n$$

such that

$$a_r < a_{r+2} \quad \text{for } 1 \leq r \leq n-2 \quad \text{and } a_r < a_{r+3} \quad \text{for } 1 \leq r \leq n-3$$

[*Note*: A *permutation* is any ordered arrangement of the numbers $1, 2, 3, \ldots, n$.]

6. AB, AC, and AD are three edges of a cube. AC is produced to E so that $AE = 2AC$, and AD is produced to F so that $AF = 3AD$. Prove that the area of the section of the cube by any plane parallel to BCD is equal to the area of the section of tetrahedron $ABEF$ by the same plane.