17-th British Mathematical Olympiad 1981

March 19, 1981

Time: 3.5 hours.

1. In a triangle ABC, H is the orthocenter and A', B', C' the midpoints of the sides BC, CA, AB, respectively. A circle centered at H intersects line $B'C'$ at D_1 and D_2, line $C'A'$ at E_1 and E_2, and line $A'B'$ at F_1 and F_2. Prove that $AD_1 = AD_2 = BE_1 = BE_2 = CF_1 = CF_2$.

2. For positive integers m, n, define

$$S_m = (n+1)-(n+1)(n+3)+(n+1)(n+2)(n+4)-(n+1)(n+2)(n+3)(n+5)+\cdots \quad (m \text{ terms}).$$

Show that $m!$ divides S_m, but that $m!(n+1)$ does not necessarily divide S_m.

3. Let a, b, c be positive numbers.
   (a) Prove that $a^3 + b^3 + c^3 \geq a^2b + b^2c + c^2a$.
   (b) Prove that $abc \geq (b + c - a)(c + a - b)(a + b - c)$.

4. In space are given n points, no four of which are coplanar. Let S be the set of tetrahedra with vertices in these points. Prove that a plane not containing any of the given points cannot cut more than $\dfrac{n^2(n-2)^2}{64}$ tetrahedra of S in a quadrilateral cross section.

5. Find, with proof, the smallest possible value of $|12^m - 5^n|$, where $m, n \in \mathbb{N}$.

6. Let $a_1, a_2, \dots, a_n$ be distinct nonzero intgers. For each $i = 1, 2, \dots, n$ we define $p_i = \prod_{j \neq i}(a_i - a_j)$. Prove that

$$\sum_{i=1}^{n} \frac{a_i^k}{p_i} \quad \text{is an integer for all } k = 0, 1, 2, \dots$$

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