18-th British Mathematical Olympiad 1982

March 18, 1982

Time: 3.5 hours.

1. Let $PQRS$ be a convex quadrilateral of area A and O be a point in its interior. Prove that if $2A = OP^2 + OQ^2 + OR^2 + OS^2$, then $PQRS$ is a square and O its center.

2. A multiple of 17 has exactly three binary ones. Prove that it has at least 6 binary zeros.

3. Consider $S_n = 1 + \frac{1}{2} + \cdots + \frac{1}{n}$, where $n \geq 2$. Prove that
$$n(n+1)^{\frac{1}{n}} - n < S_n < n - (n-1)n^{-\frac{1}{n-1}}.$$

4. A sequence of real numbers $u_1, u_2, u_3, \dots$ is determined by u_1 and
$$u_{n+1}^3 = u_n + \frac{15}{64} \quad \text{for } n \geq 1.$$
By considering the curve $x^3 = y + \frac{15}{64}$ or otherwise, describe, with proof, the behavior of u_n as $n \to \infty$.

5. A right circular cone of radius r is given. The distance from the vertex V to a point on the perimeter of the base circle is 1. A plane section of the cone is an ellipse with the lowest point L and the highest point H. Two paths from L to H are marked on the surface of the cone, on one side of plane VHL: path R_1 going along the ellipse, and the shortest path R_2. Find the condition under which R_1 and R_2 meet between L and H.

6. Show that the number of sequences $a_1a_2\dots a_n$ of 0 and 1 with exactly m pairs 01 is $\binom{n+1}{2m+1}$.

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