13-rd Mathematical Olympiad 1977

January 1977

Time allowed: 3.5 hours

1. A non-negative integer $f(n)$ is assigned to each positive integer n in such a way that the following conditions are satisfied:

   (a) $f(mn) = f(m) + f(n)$, for all positive integers m, n;

   (b) $f(n) = 0$ whenever the units digit of n (in base 10) is a 3; and

   (c) $f(10) = 0$.

   Prove that $f(n) = 0$, for all positive integers n.

2. The sides BC, CA, and AB of a triangle touch a circle at X, Y, and Z respectively. Prove that the centre of the circle lies on the straight line through the midpoints of BC and of AX.

3. (a) Prove that if x, y, and z are non-negative real numbers, then

$$x(x-y)(x-z) + y(y-z)(y-x) + z(z-x)(z-y) \geq 0.$$

   (b) Hence or otherwise, show that, for all real numbers a, b, and c,

$$a^6 + b^6 + c^6 + 3a^2b^2c^2 \geq 2\left(b^3c^3 + c^3a^3 + a^3b^3\right)$$

4. The equation $x^3 + qx + r = 0$, where $r \neq 0$, has roots u, v, and w. Express the roots of the equation $r^2x^3 + q^3x + q^3 = 0$ in terms of u, v, and w. Show that if u, v, and w are real, then this latter equation has no root in the interval $-1 < x < 3$.

5. The regular pentagon $A_1A_2A_3A_4A_5$ has sides of length $2a$. For each $i = 1, 2, \ldots, 5$, K_i is the sphere with centre A_i and radius a. The spheres $K_1, K_2, \ldots, K_5$ are all touched externally by each of the spheres P_2 and P_2 also of radius a. Determine, with proof, whether or not P_1 and P_2 have a common point.

6. The polynomial $26(x+x^2+x^3+\cdots+x^n)$, where $n > 1$, is to be decomposed into a sum of polynomials, not necessarily all different. Each of these polynomials is to be of the form $a_1x + a_2x^2 + a_3x^3 + \cdots + a_nx^n$, where each a_i is one of the numbers $1, 2, 3, \ldots, n$ and no two a_i are equal. Find all the values of n for which this decomposition is possible.