14-th Mathematical Olympiad 1978

January 1978

Time allowed: 3.5 hours

1. Determine, with proof, the point P inside a given triangle ABC for which the product $PL \cdot PM \cdot PN$ is a maximum, where L, M, and N are the feet of the perpendiculars from P to BC, CA, and AB respectively.

2. Prove that there is no proper fraction m/n with denominator $n \leq 100$, the decimal expansion of which contains the block of consecutive digits 167, in that order.

3. Show that there is one and only one sequence $\{u_n\}$ of integers such that

$$u_1 = 1, \quad u_1 < u_2, \quad \text{and } u_n^3 + 1 = u_{n-1}u_{n+1} \quad \text{for all } n > 1$$

4. An *altitude* of a tetrahedron is a straight line through a vertex which is perpendicular to the opposite face. Prove that the four altitudes of a tetrahedron are concurrent if and only if each edge of the tetrahedron is perpendicular to its opposite edge.

5. Inside a cube of side 15 units there are 11000 given points. Prove that there exists a sphere of unit radius containing at least six of the given points.

6. Show that if n is a non-zero integer, $2\cos n\theta$ is a polynomial of degree n in $2cos\theta$. Hence or otherwise, prove that if k is rational, then $\cos k\pi$ is either equal to one of the numbers $0, \pm\frac{1}{2}$, or ± 1, or is irrational.