15-th Mathematical Olympiad 1979

January 1979

Time allowed: 3.5 hours

1. Find all triangles ABC for which $AB + AC = 2$ and $AD + BD = \sqrt{5}$, where AD is the altitude through A, meeting BC at the point D.

2. From a point O in three-dimensional space three given rays, OA, OB, and OC, emerge, with $\angle BOC = \alpha$, $\angle COA = \beta$, and $\angle AOB = \gamma$, $0 < \alpha, \beta, \gamma < \pi$. Prove that, given $2s > 0$, there are unique points X, Y, and Z on OA, OB and OC respectively such that the triangles YOZ, ZOX, and XOY have the same perimeter $2s$. Express OX in terms of s, $\sin(\alpha/2)$, $\sin(\beta/2)$, and $\sin(\gamma/2)$.

3. $S = \{a_1, a_2, \ldots, a_n\}$ is a set of distinct positive odd integers. The differences $|a_i - a_j|$ $(1 < i \leq j < n)$ are all distinct. Prove that

$$\sum_{i=1}^{n} \geq \frac{1}{3} n \left(n^2 + 2\right)$$

4. The function f is defined on the rational numbers and takes only rational values. For all rationals x and y,

$$f(x + f(y)) = f(x)f(y)$$

   Prove that f is a constant function.

5. If n is a positive integer, denote by $p(n)$ the number of ways of expressing n as the sum of one or more positive integers. Thus $p(4) = 5$, as there are five different ways of expressing 4 in terms of positive integers; namely,

$$1+1+1+1, \quad 1+1+2, \quad 1+3, \quad 2+2, \quad \text{and } 4$$

   Prove that $p(n+1) - 2p(n) + p(n-1) \geq 0$ for each $n > 1$.

6. Prove that there are no prime numbers in the infinite sequence

$$10001, \quad 100010001, \quad 1000100010001, \ldots$$