11-st Mathematical Olympiad 1975

January 1975

Time allowed: 3 hours

1. Given that x is a positive integer, find (with proof) all solutions of

$$\left[\sqrt[3]{1}\right] + \left[\sqrt[3]{2}\right] + \cdots + \left[\sqrt[3]{(x^3-1)}\right] = 400$$

[*Note*: $[z]$ denotes the largest integer $\leq z$.]

2. The first n prime numbers $2, 3, 5, \ldots, p_n$ are partitioned into two disjoint sets A and B. The primes in A are $a_1, a_2, \ldots, a_h$ and the primes in B are $b_1, b_2, \ldots, b_k$, where $h + k = n$. The two products

$$\prod_{i=1}^{h} a_i^{\alpha_i} \quad \text{and} \quad \prod_{i=1}^{k} b_i^{\beta_i}$$

are formed, where the α_i and β_i are any positive integers. If d divides the difference of these two products, prove that either $d = 1$ or $d > p_n$.

3. Use the pigeonhole principle to solve the following problem. Given a point O in the plane, the disc S with centre O and radius 1 is defined as the set of all points P in the plane such that $|OP| \leq 1$, where $|OP|$ is the distance of P from O. Prove that if S contains seven points such that the distance from any one of the seven points to any other is ≥ 1, then one of the seven points must be at O.

[*Note*: The pigeonhole principle states that if more than n objects are put into n pigeonholes, some pigeonhole must contain more than one object.]

4. In a triangle ABC, three parallel lines AD, BE, and CF are drawn, meeting the sides BC, CA, and AB in D, E, and F respectively. The points P, Q, and R are collinear and divide AD, BE, and CF respectively in the same ratio $k : 1$. Find the value of k.

5. Let m be a fixed positive integer. You are given that

$$\binom{2m}{0} + \binom{2m}{1}\cos\theta + \binom{2m}{2}\cos 2\theta + \cdots + \binom{2m}{m}\cos 2m\theta = \left(2\cos\frac{\theta}{2}\right)^{2m}\cos m\theta$$

where there are $2m+1$ terms on the LHS; the value of each side of this identity is defined to be $f_m(\theta)$. The function $g_m(\theta)$ is defined by

$$g_m(\theta) = \binom{2m}{0} + \binom{2m}{2}\cos 2\theta + \binom{2m}{4}\cos 4\theta + \cdots + \binom{2m}{2m}\cos 2m\theta$$

Given that there is no rational k for which $\alpha = k\pi$, find the values of α for which

$$\lim_{m\to\infty} \frac{g_m(\alpha)}{f_m(\alpha)} = \frac{1}{2}$$

6. Prove that, if n is a positive integer greater than 1, and $x > y > 1$, then

$$\frac{x^{n+1}-1}{x\left(x^{n-1}-1\right)} > \frac{y^{n+1}-1}{y\left(y^{n-1}-1\right)}$$

7. Prove that there is only one set of real numbers $x_1, \ldots, x_n$ such that

$$\left(1-x_1\right)^2+\left(x_1-x_2\right)^2+\cdots+\left(x_{n-1}-x_n\right)^2+x_n^2=\frac{1}{n+1}$$

8. The interior of a wine glass is a right circular cone. The glass is half filled with water and is then slowly tilted so that the water reaches a point P on the rim. If the glass is further tilted (so that water spills out), what fraction of the conical interior is occupied by water when the horizontal plane of the water level bisects the generator of the cone furthest from P?