# 12-nd Mathematical Olympiad 1976

*January 1976*

Time allowed: 3.5 hours

1. Find, with proof, the length d of the shortest straight line which bisects the area of an arbitrary given triangle, but which does not pass through any of the vertices. Express $d$ in terms of the area $\Delta$ of the triangle and one of its angles. Show that there is always a shorter line (not straight) which bisects the area of the given triangle.

2. Prove that if $x$, $y$, and $z$ are positive real numbers, then
$$\frac{x}{y+z}+\frac{y}{z+x}+\frac{z}{x+y}\geq\frac{3}{2}$$

3. $S_1, S_2, \ldots, S_{50}$ are subsets of a finite set $E$. Each subset contains more than half the elements of $E$. Show that it is possible to find a subset $F$ of $E$ having not more than five elements, such that each $S_i$ $(1 \leq i \leq 50)$ has an element in common with $F$.

4. Prove that if $n$ is a non-negative integer, then $19 \cdot 8^n + 17$ is not a prime number.

5. Prove that, if $\alpha$ and $\beta$ are real numbers, and $r$ and $n$ are positive integers with $r$ odd and $r \leq n$, then
$$\sum_{t=0}^{(r-1)/2}\binom{n}{r-t}\binom{n}{t}(\alpha\beta)^t\left(\alpha^{r-2t}+\beta^{r-2t}\right)=\sum_{t=0}^{(r-1)/2}\binom{n}{r-t}\binom{r-t}{t}(\alpha\beta)^t(\alpha+\beta)^{r-2t}$$
[*Note*: $\binom{m}{s}$ denotes the coefficient of $x^s$ in the expansion of $(1+x)^m$.]

6. A sphere with centre $O$ and radius $r$ is cut by a horizontal plane distance $r/2$ above $O$ in a circle $K$. The part of the sphere above the plane is removed and replaced by a right circular cone having $K$ as its base and having its vertex $V$ at a distance $2r$ vertically above $O$. $Q$ is a point on the sphere on the same horizontal level as $O$. The plane $OVQ$ cuts the circle $K$ in two points, $X$ and $Y$, of which $Y$ is the further from $Q$. $P$ is a point of the cone lying on $VY$, the position of which can be determined by the fact that the shortest path from $P$ to $Q$ over the surfaces of cone and sphere cuts the circle $K$ at an angle of 45°. Prove that
$$VP=\frac{\sqrt{3}r}{1+\frac{1}{\sqrt{5}}}$$

*Note*: In a spherical triangle $ABC$, the sides are arcs of great circles, centre $O$, and the sides are measured by the angles that they subtend at $O$. You may find the following spherical triangle formulae useful:

$$\frac{\sin a}{\sin A} = \frac{\sin b}{\sin B} = \frac{\sin c}{\sin C}$$

$$\cos a = \cos b \cos c + \sin b \sin c \cos A$$