7-th Mathematical Olympiad 1971

January 1971

Time allowed: 3 hours

1. (a) Factorize $(x+y)^7 - (x^7 + y^7)$.

   (b) Prove that there is no integer n for which $2n^3 + 2n^2 + 2n + 1$ is a multiple of 3.

2. $x_1 = 9$ and $x_{r+1} = 9^{x_r}$ for all positive integral r. Prove that the last two digits of x_3 written to base 10 are the same as the last two digits of x_4 written to base 10. What are these two digits?

3. Of a regular polygon of $2n$ sides there are n diagonals which pass through the centre of the inscribed circle. The angles which these diagonals subtend at two given points A and B on the circumference are $a_1, a_2, a_3, \ldots, a_n$ and $b_1, b_2, b_3, \ldots, b_n$. Prove that

$$\sum_{i=1}^{n} \tan^2 a_i = \sum_{i=1}^{n} \tan^2 b_i$$

4. Given a set of $n+1$ positive integers, none of which exceeds $2n$, prove by induction or otherwise that at least one member of the set must divide another member of the set.

5. The triangle ABC has angles A, B, and C, in descending order of magnitude. Circles are drawn such that each circle cuts each side of the triangle internally in two real distinct points. The lower limit to the radii of such circles is the radius of the incircle of the triangle ABC. Show that the upper limit is not R, the radius of the circumcircle, and find this upper limit in terms of R, A, and B.

6. (a) $I(x) = \int\limits_c^x f(x,u)du$, where c is a constant. Show why

$$I'(x) = f(x,x) + \int\limits_c^x \frac{\partial f}{\partial x} du.$$

   (b) Find $\lim_{\theta \to 0} \cot\theta \sin(t \sin\theta)$.

   (c) $G(t) = \int_0^t \cot\theta \sin(t\sin\theta) d\theta$. Prove that $G'(\frac{\pi}{2}) = \frac{2}{\pi}$.

7. Two real numbers h and k are given, such that $h > k > 0$. Find the probability that two points chosen at random on a straight line of length h should be at a distance of less than k apart.

8. A is a 3×2 matrix and B is a 2×3 matrix. The elements of each matrix are numbers. $AB = M$ and $BA = N$. $\det M = 0$ and $\det N \neq 0$. Also, $M^2 = kM$, where k is a number. Determine $\det N$ in terms of k, proving your result. You may assume the usual rules for combining matrices.

   [*Note*: k is not a matrix; kC is defined as the matrix each element of which is k times the corresponding element of C.]

9. Two uniform solid spheres of equal radii are so placed that one is directly above the other. The bottom sphere is fixed and the top sphere, initially at rest, rolls off. If the coefficient of friction between the two spherical surfaces is μ, show that slipping occurs when $2\sin\theta = \mu(17\cos\theta - 10)$, where θ is the angle that the line of centres makes with the vertical.

   [*Note*: The moment of inertia of a solid sphere of radius r and mass M about a diameter is $\frac{2}{5}Mr^2$.]