8-th Mathematical Olympiad 1972

January 1972

Time allowed: 3 hours

1. S is a set of elements $\{a, b, \dots\}$ each pair of which are distinct. R is a relation holding or not holding between every ordered pair of distinct elements of S according to the following rules:

   (a) either aRb or bRa, but not both;

   (b) if aRb and bRc, then cRa.

   Find the largest number of elements that S can have, proving your result.

2. Let a, b, c, and d be integers, with $a \neq b$. Show that there are at most four different points on the hyperbola

   $$(x + ay + c)(x + by + d) = 2$$

   each of the coordinates of which are integers. Find a set of conditions which are both necessary and sufficient for there to be four such points.

3. In a plane, two circles of unequal radii intersect at A and B, and through an arbitrary point P a straight line L is to be constructed so that the two circles intersect equal chords on L. By considering distances from the point of intersection of L with AB (produced if necessary), or otherwise:

   (a) find a *ruler and compass* construction for L;

   (b) state the region of the plane in which P must lie for the construction to be possible.

4. A point P is on a smooth curve in a plane containing two fixed points A and B; $PA = a$, $PB = b$, and $AB = c$. The angle $APB = \theta$, where $c^2 = a^2 + b^2 - 2ab\cos\theta$. Show why

   $$\sin^2\theta(ds)^2 = (da)^2 + (db)^2 - 2(da)(db)\cos\theta,$$

   where s measures the distance along the curve. The point P moves along the curve so that at time t during the interval $\frac{1}{2}T < t < T$ its distance from A and B are given, respectively, by $h\cos(\frac{t}{T})$ and $k\sin(\frac{t}{T})$, where h, k, and T are positive constants. Prove that the speed of P during this interval varies as $\operatorname{cosec}\theta$.

5. In a right circular cone the semi-vertical angle of which is θ, a cube is placed so that four of its vertices are on the base and four on the curved surface. Prove that as θ varies the maximum value of the ratio of the volume of the cube to the volume of the cone occurs when $\sin\theta = \frac{1}{3}$.

6. $T_k = k - 1$, for $k = 1, 2, 3, 4$. For all $k \geq 3$, $T_{2k-1} = T_{2k-2} + 2^{k-2}$ and $T_{2k} = T_{2k-5} + 2^k$. Show that, for all $k > 0$,

$$1 + T_{2k-1} = \left[\frac{12}{7} 2^{k-1}\right] \qquad \text{and} \qquad 1 + T_{2k} = \left[\frac{17}{7} 2^{k-1}\right]$$

where $[x]$ denotes the greatest integer not exceeding x.

7. Starting from $p_1 = 2$, $q_1 = 1$, $r_1 = 5$, and $s_1 = 3$, sequences of integers are formed by using the following relations, which hold for all positive integral n:

$$p_{n+1} = p_n^2 + 3q_n^2 \quad , \quad q_{n+1} = 2p_n q_n \quad , \quad r_n = p_n + 3q_n \quad , \quad s_n = p_n + q_n$$

Prove that:

(a) $p_n/q_n > \sqrt{3} > r_n/s_n$

(b) p_n/q_n differs from $\sqrt{3}$ by less than $s_n/(2r_n q_n^2)$.

8. Three children, A, B, and C, build cairns of stones on a beach and try to knock down each other's cairns. Each time a child aims at a cairn, he has a fixed chance of hitting it: A has a chance of $\frac{3}{4}$ of hitting the cairn at which he aims, B a chance of $\frac{2}{3}$ and C a chance of $\frac{1}{2}$. Shots aimed at one cairn do not hit another. A player whose cairn is hit is out of the game. A game is won by being the only remaining player. The rule of play is for the players in the game to throw simultaneously, choosing with equal probability between his two opponents' cairns (when there are two). Find the probability that C wins, with A being eliminated in the first round.

9. A rocket, free of gravitational forces, is accelerating in a straight line. Its mass is M and that of its fuel is $m = m_0 e^{-k}$. The relative velocity of escape of the fuel is $v = v_0 e^{-kt}$. If m_0 is small compared with M, show that the terminal velocity of the rocket is approximately $\frac{m_0 v_0}{2M}$ greater than its initial velocity.