9-th Mathematical Olympiad 1973

January 1973

Time allowed: 3 hours

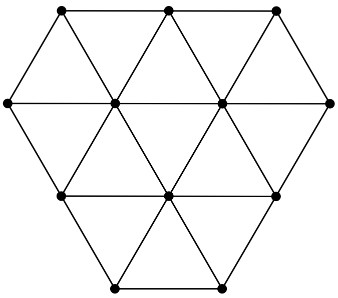
1. (a) Two fixed circles are touched by a variable circle at P and Q. Prove that PQ passes through one of two fixed points.

   (b) State a true theorem about ellipses (or if you like about conics in general) of which the result in part (a) is a particular case.

2. Nine points are given in the interior of the unit square. Prove that there exists a triangle of area $\leq \frac{1}{8}$ the vertices of which are three of the points.

3. A curve consisting of the quarter circle $x^2 + y^2 = r^2$, $x, y \geq 0$, together with the line segment $x = r$, $-h \leq y \leq 0$, is rotated about $x = 0$ to form a surface of revolution which is a hemisphere on a cylinder. A string is stretched tightly over the surface from the point on the curve $(r \sin\theta, r\cos\theta)$ to the point $(-r, -h)$ in the plane of the curve. Show that the string does not lie in a plane if $\tan\theta > r/h$. [*Note*: You may assume spherical triangle formulae such as $\cos a = \cos b \cos c + \sin b \sin c \cos A$, or $\sin A \cot B = \sin c \cot b - \cos c \cos A$. In a spherical triangle the sides a, b, and c are arcs of great circles and are measured by the angles that they subtend at the centre of the sphere.]

4. You have a large number of congruent equilateral triangles on a table and you want to fit n of these together to make a convex equiangular hexagon (that is, one the interior angles of which are each 120°). Obviously, n cannot be any positive integer. The smallest n is 6, the next smallest is 10, and the next 13. Determine conditions for possible n.



$n = 13$

5. There is an infinite set of positive integers of the form $2^n - 3$ with the following property Q:

   no two members of the set have a common prime factor.

   Here is an outline of a proof of this fact.

Suppose that there is a finite set $S = \{2^{m_1}-3, 2^{m_2}-3, \ldots, 2^{m_k}-3\}$ with property Q having k members. Let the prime factors of these k numbers be $p_1, p_2, \ldots, p_t$. Consider the number $N = 2^{(p_1-1)(p_2-1)\cdots(p_t-1)+1}$. By Fermat's Theorem, $a^{p-1} \equiv 1 \pmod p$ for every prime p that does not divide a. Hence $N - 3 \equiv -1 \pmod{p_r}$, for each $r (1 \le r \le t)$, and $N-3$ may be added to S to give a larger set with property Q.

Give a properly expanded and reasoned proof that there is an infinite set of positive integers of the form $2^n - 7$ with property Q.

6. In answering general knowledge questions (framed so that each question is answered *yes* or *no*) the teacher's probability of being correct is α and a pupil's probability of being correct is β or γ according as the pupil is a boy or a girl. The probability of a randomly chosen pupil agreeing with the teacher's answer is $\frac{1}{2}$. Find the ratio of the number of boys to girls in the class.

7. The life-table issued by the Registrar-General of Draconia shows, out of each 10000 live births, the number y expected to be alive x years later. When $x = 60$, $y = 4820$. When $x = 80$, $y = 3205$. At age 100, all Draconians are put to death. For $60 \le x \le 100$ the curve $y = Ax(100 - x) + B/(x-40)^2$ fits the figures in the table very closely, A and B being constants. Determine the life-expectancy (in years correct to one decimal place) of a Draconian aged 70.

8. Call the matrix $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ the companion matrix for the mapping

$$T : z \longrightarrow \frac{az+b}{cz+d}$$

   (a) Prove that, if M_i is the companion matrix for $T_i, (i = 1, 2)$, then $M_1 M_2$ is the companion matrix for the mapping $T_1 T_2$.
   (b) Find conditions on a, b, c, and d so that $T^4 = I$ but $T^2 \ne I$.

9. Let L_r equal the determinant

$$\begin{vmatrix} x & y & 1 \\ a + c\cos\theta_r & b + c\sin\theta_r & 1 \\ l + n\cos\theta_r & m + n\sin\theta_r & 1 \end{vmatrix}$$

   Show that the lines $L_r = 0, r = 1, 2, 3$, are concurrent and find the coordinates of the point at which they meet.

10. Construct a detailed flowchart for a computer program to print out all positive integers up to 100 of the form $a^2 - b^2 - c^2$, where a, b, and c are positive integers and $a \ge b + c$. (There is no need to print in ascending order or to avoid repetitions.)

11. (a) Two uniform rough right circular cylinders, A and B, with the same length, have radii a and b and masses M and m respectively. Cylinder A rests with a generator in contact with a rough horizontal table. Cylinder B rests on A, initially in unstable equilibrium, with its axis vertically above that of A. Equilibrium is disturbed, B rolls on A, and A rolls on the table. In the subsequent motion, the plane containing the axes makes an angle θ with the vertical. Draw diagrams showing angles, forces, etc., for the period during which there is no slipping. Write down equations which will give on elimination a differential equation for θ, stating the principles used. Indicate how the elimination could be done (you are not asked to do it).

(b) Such a differential equation is, with $k = M/m$,

$$\ddot{\theta}(4 + 2\cos\theta - 2\cos^2\theta + 9k/2) + \dot{\theta}\sin\theta(2\cos\theta - 1) = \frac{3g(1+k)\sin\theta}{a+b}$$

Obtain $\dot{\theta}$ in terms of θ.

[*Note*: The moment of inertia of a uniform cylinder about its axis is $\frac{1}{2}(\text{mass}) \times (\text{radius})^2$.]