6-th Mathematical Olympiad 1970

January 1970

Time allowed: 3 hours

1. (a) Express the finite series

$$\frac{1}{\log_2 a} + \frac{1}{\log_3 a} + \cdots + \frac{1}{\log_n a}$$

as a quotient of logarithms to base 2. Does the series converge as $n \longrightarrow \infty$?

(b) Evaluate the sum of the coefficients of the polynomial

$$p(x) = (1 + x - x^2)^3(1 - 3x + x^2)^2$$

and the sum of the coefficients of its derivative.

2. Sketch the curve the equation of which is given by $x^2 + 3xy + 2y^2 + 6x + +12y+4 = 0$. About which point does it possess some symmetry property?

3. You are probably aware of the remarkable fact that the three points of intersection of three appropriate pairs of trisectors of the internal angles y of any triangle T form the vertices of an equilateral triangle t. Given a real constant $A > 0$, consider the class of all triangles T the area of which equals A. Find, with this class, the maximum value of the area of the triangle t. Discuss whether there is a minimum value.

4. S is any set of n positive integers the sum of which is not divisible by n. Prove that it is always possible to choose a subset of S, the sum of the members of which is divisible by n.

5. A cube of wood is to be sawn up into at least 300 separate pieces. What is the minimum number of sawcuts which need to be made? (Each sawcut is planar, and it is assumed that pieces are not moved relative to each other -so that the block remains intact as a cube- until after the last cut has been made.)

6. In the region $|x| \leq a$, $y(x)$ is defined by the differential equation $dy/dx = f(x)$, where $f(x)$ is a given even, continuous function of x. Prove that:

   (a) $y(-a) - 2y(0) + y(a) = 0$;

   (b) $\int_{-a}^{a} y(x)dx = 2ay(0)$.

   A numerical step-by-step solution is calculated starting from the point $(-a, 0)$ using $2N$ equal x-wise steps, according to the usual scheme $g(x_{n+1}) = g(x_n) + \delta x \cdot g'(x_n)$. Prove that in general this solution will not satisfy the result (a) above. Suggest a modified step-by-step procedure which would ensure that (a) is satisfied.

7. The two base angles B and C of an isosceles triangle ABC are equal to $50°$. The point D lies on BC, so that $\angle BAD = 50°$, and the point E lies on AC, so that $\angle ABE = 30°$. Find $\angle BED$.

8. Eight electric light bulbs are arranged in a row, each controlled by its own on/off switch. State how many different patterns (or *states*) of lit bulbs there are (including the pattern of all bulbs off). Denote the N distinct states by S_i, $i = 1, 2, 3, \ldots, N$. Let n_j be the number of switches which have to be altered to change from S_i to S_{i+1} (with the convention that $SN + 1 = S_1$). Find the smallest possible value of $n = \sum_{i=1}^{n} n_i$; that is, find the smallest value for the total number of switch-alterations which have to be made so as to run through all the possible states of the lights in some order, before returning to the initial state.

9. Find rational numbers x and y such that

$$\sqrt{2\sqrt{3} - 3} = \sqrt[4]{x} - \sqrt[4]{y}$$

10. Unlimited supplies of straight rods (of negligible thickness) are available, the length of each rod being an integral multiple of 1 m. The rods are laid out on a large horizontal surface to form as many separate right-angled triangles as possible. Let $N(i)$ denote the number of different right- angled triangles the shortest side of which is a rod of length i metres. (For values of i for which no such right-angled triangle exists, we define $N(i) = 0$.)

    (a) Find some kind of *formula* for $N(i)$.

    (b) Prove that, for any given positive integer M, there exists a value of i such that $N(i) > M$.

    (c) Discuss whether $N(i) \longrightarrow \infty$ as $i \longrightarrow \infty$.