5-th Mathematical Olympiad 1969

January 1969

Time allowed: 3 hours

1. Find the condition on the distinct real numbers a, b, and c such that $(x-a)(x-b)/(x-c)$ takes all real values for real values of x. Sketch two graphs of $(x-a)(x-b)/(x-c)$ to illustrate:
   (a) a case in which the condition is satisfied; and
   (b) a case in which the condition is not satisfied.
2. Find all the real solutions of $\cos x + \cos^5 x + \cos 7x = 3$.
3. A square piece of plywood is cut up into n^2 equal square pieces. These are then rearranged to make four rectangles with one square piece left over, in such a way that the nine dimensions of the five shapes (including the edge length of the square piece) are all different. Find the minimum value of n for this situation to be possible, and specify the shapes of the four rectangles for this value of n. State all other values of n for which the situation is possible, justifying your answer.
4. Find all possible pairs (a, b) of integers satisfying $a^2 - 3ab - a + b = 0$.
5. A long corridor of unit width has a right-angled corner in it. A rigid length of pipe (the thickness of which may be neglected) lies on, and is everywhere in contact with, the plane floor of the corridor. The length of the pipe (which may be curved) is defined as the straight-line distance between its two ends. Find the maximum length of pipe subject to the condition that it can be moved along both arms of the corridor and round the corner without leaving contact with the floor.
6. Prove that, if a, b, c, d, and e are positive integers, any common factor of $(ae + b)$ and $(ce + d)$ is also a factor of $(ad - bc)$.
7. Let $f(x)$ denote a real-valued function of the real variable x, not identically zero, and differentiable at $x = 0$.
   (a) If $f(x)\cdot f(y) = f(x+y)$ for all real values of x and y, prove that $f(x)$ is differentiable any number of times for all x, and that $\sum\limits_{i=0}^{\infty} f(i) = \frac{1}{1-f(1)}$ when $f(1) < 1$.
   (b) If $f(x) \cdot f(y) + f(x - y)$ for all real values of x and y, find $f(x)$.
8. A square has sides of length x. All four vertices of the square lie on the sides of a circumscribing triangle. The incircle of the triangle has radius r. Prove that $2r > x > r\sqrt{2}$.

9. Let $P(n)$ be the proposition that *a polygon of* n^2 *sides can be drawn so that its vertices lie at the* n^2 *points of a square* $n \times n$ *array.* State, with proof, the truth or otherwise of $P(4)$ and $P(5)$. Formulate, with some justification, a conjecture about the truth of $P(n)$ for $n > 6$. [*Note*: A polygon satisfies the following common-sense properties: (a) no two sides coincide; (b) no two sides intersectunless they are adjacent, non-parallel sides, in which case their intersection is one of the polygon's vertices.]

10. Describe a ruler-and-compass method of constructing an equilateral triangle the area of which equals that of a given triangle.