# 4-th Mathematical Olympiad 1968

*January 1968*

Time allowed: 3 hours

1. A circle $C$ of unit radius rolls without slipping along the outside of the circle with centre the origin and radius 2 in the $(x, y)$-plane. A fixed point $P$ of $C$ is originally at the position $(2, 0)$. Find equations, giving the coordinates $(x, y)$ of $P$ in terms of a suitable parameter $B$, as $C$ rotates. Sketch the locus $S$ described by $P$, showing all tangents parallel to the $x$- and $y$-axes.

2. Cows are put out to pasture when the grass has reached a certain height. Thereafter, as the cows eat the grass, the grass continues to grow as the pasture is consumed. If 15 cows can consume the grass in three acres of pasture in four days, while 32 cows can consume the grass in four acres in two days, how many cows will be required to consume the grass in six acres in three days?

3. The *distance* $d$ between two points $(x_1, y_1)$ and $(x_2, y_2)$ in the $(x, y)$-plane is defined by
$$d = |x_2 - x_1| + |y_2 - y_1|$$
Using this notion of distance, find the locus of all points $(x, y)$ satisfying $x \geq 0$, $y \geq 0$, and which are equidistant from the origin and from a fixed point $(a, b)$ in the plane, where $a > b$. Distinguish the cases according to the signs of $a$ and $b$.

4. Two spheres of radii $a$ and $b$ are tangent to each other and a plane is tangent to these spheres at different points. Find the radius of the largest sphere which can pass between the first two spheres and the plane.

5. Given $x$, $y$, and $z$ such that
$$\sin x + \sin y + \sin z = 0$$
and
$$\cos x + \cos y + \cos z = 0,$$
prove that
$$\sin 2x + \sin 2y + \sin 2z = 0$$
and
$$\cos 2x + \cos 2y + \cos 2z = 0$$

6. If $a_1, a_2 \ldots, a_7$ are integers, and $b_1, b_2 \ldots, b_7$ are the same integers rearranged, show that the value of
$$(a_1 - b_1)(a_2 - b_2) \cdots (a_7 - b_7)$$
is necessarily even.

7. A knock-out ping-pong tournament is arranged among n people and a new ball is used for each game. How many balls are needed?

8. A chord of length $\ell$ divides the interior of a circle of radius $r$ into two regions, $D_1$, and $D_2$. A circle $S$ of maximal radius is inscribed in $D_1$; the area of the part of $D_1$ outside $S$ is $A$. Show that $A$ is greatest when the area of $D_1$ exceeds that of $D_2$, and when

$$\ell = \frac{16\pi r}{16 + \pi^2}$$

9. Find the lengths of the sides of a triangle if its altitudes have lengths 3, 4, and 6.

10. The faces of a tetrahedron are formed by four congruent triangles. If a is the angle between a pair of opposite edges of the tetrahedron, show that

$$\cos\alpha = \frac{\sin(B - C)}{\sin(B + C)}$$

where $B$ and $C$ are the angles adjacent to one of these edges in a face of the tetrahedron.

11. The sum of the reciprocals of a set of $n$ different positive integers is equal to one. If $n = 3$ show that there is only one such set and find it. Find such a set for $n = 4, 5$, and more generally for any value of $n > 3$.

12. Find, with proof, the maximum number of points which can be placed on the surface of a sphere of unit radius such that the distance between any two of the points is:

    (a) at least $\sqrt{2}$;
    
    (b) greater than $\sqrt{2}$.