

3-rd Mathematical Olympiad 1967

January 1967

Time allowed: 3 hours

1. If α and β are the roots of $x^2 + px + 1 = 0$, and γ and δ are the roots of $x^2 + qx + 1 = 0$, show that

$$(\alpha - \gamma)(\beta - \gamma)(\alpha + \delta)(\beta + \delta) = q^2 - p^2$$

2. By putting $y = tx$, or otherwise, draw the graph of $x^8 + xy + y^8 = 0$. Show clearly all turning values of x and y , and the behaviour of the graph when x and y become large.
3. (a) The lengths a and b of two sides of a triangle satisfy $a > b$. The lengths of the corresponding altitudes are h_a and h_b . Prove that $a + h_a \geq b + h_b$. Discuss all cases in which equality occurs.
- (b) If the circumcentre and the incentre of a triangle Δ coincide, prove that Δ is equilateral.
4. When is it possible to find points equidistant both from two given points P and Q and from a given straight line AB ? When it is possible, show how to construct such points with ruler and compasses only.
5. Show that $f(t) = (t - \sin t)(\pi - t - \sin t)$ is increasing in the interval $0 < t < \pi/2$.
6. Find all real numbers x in the interval $0 \leq x \leq 2\pi$ such that

$$2 \cos x \leq \left| \sqrt{1 + \sin 2x} - \sqrt{1 - \sin 2x} \right| \leq \sqrt{2}$$

[Note: If t is any positive number, $\sqrt{t}$ is the positive square root of t . If x is any real number, $|x| = \sqrt{x^2}$.]

7. Four real numbers x_1, x_2, x_3 , and x_4 are such that the sum of any one of them with the product of the other three is equal to 2. Find all possible solutions.
8. Find all non-negative integers n such that $5^n - 4^n$ is divisible by 61.
9. If α, β , and γ are the angles of a triangle with $\alpha\beta\gamma \neq 0$, show that

$$\cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma \geq \frac{3}{4}$$

Find all such triangles for which equality holds. Show also that $\cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma$ does not have a greatest value.

10. Alan, Brian, Colin, and David collect stamps. Alan collects British stamps issued before 1900 and foreign stamps. Brian collects British stamps issued after 1900 and foreign special issues. Colin collects foreign stamps issued before 1900 and British special issues. David collects foreign stamps issued after 1900 and British special issues. Are there any stamps which no-one collects? Are there any stamps which everyone collects? What stamps could David give away that would be of interest to Alan but not to Brian?
11. In a certain town, the streets are arranged in a rectangular grid. A man wishes to go from one place to another place m streets east and n streets north. How many shortest paths are there?