2-nd Mathematical Olympiad 1966

January 1966

Time allowed: 3 hours

1. Find the least and the greatest values of

$$\frac{x^4 + x^2 + 5}{\left(x^2 + 1\right)^2}$$

for real values of x.

2. Give the conditions under which all the roots of the equations

$$\pm\sqrt{x-a} \pm \sqrt{x-b} \pm \sqrt{x-c} = 0$$

are real, where a, b, and c are distinct real numbers.

3. Sketch the graph of $y^2 = \frac{x^2(x+1)}{x-1}$. Find all turning values of y, and describe the behaviour of the graph when x and y become large.

4. The points A, B, C, and D are four consecutive vertices of a regular polygon. If

$$\frac{1}{AB} = \frac{1}{AC} + \frac{1}{AD}$$

how many sides must the polygon have?

5. A square nut of side a is to be turned by means of a spanner, the hole in which consists of a regular hexagon of side b. Find the conditions on a and b for this to be possible.

6. Find the largest interval of values x for which the expression

$$y = \sqrt{x-1} + \sqrt{x + 24 - 10\sqrt{x-1}}$$

has a constant value. [*Note*: $\sqrt{t}$ is defined for $t \geq 0$ to be that non-negative number whose square is t.]

7. Prove that $\sqrt{2}$, $\sqrt{3}$, and $\sqrt{5}$ cannot be terms of the same arithmetic progression.

8. The faces of a cube are coloured by six colours, so that each face has a different colour. Find how many cubes of distinct appearance can be produced in this way. Show also that 1680 regular octahedra of distinct appearance can be produced by colouring the eight faces of a regular octahedron with eight given colours so that each face has a different colour.

9. If α, β, and γ are the angles of a triangle, find:

   (a) the smallest possible value of $\tan(\alpha/2) + \tan(\beta/2) + \tan(\gamma/2)$;

   (b) the largest possible value of $\tan(\alpha/2)\tan(\beta/2)\tan(\gamma/2)$.

10. One hundred students, all of different heights, are arranged in a square of 10 rows by 10 columns. In each row the tallest student is selected, and the shortest of these tall students is labelled A. In each column the shortest student is selected, and the tallest of these short students is labelled B. If A and B are different persons, find which of them is the taller and why.

11. (a) prove that, among any 52 integers, there must always exist two integers such that either the sum or the difference of these two integers is divisible by 100.
    (b) Prove that, given. any 100 integers, none of which is divisible by 100, it is possible to find two or more of these integers the sum of which is divisible by 100.