1-st Mathematical Olympiad 1965

January 1965

Time allowed: 3 hours

1. Sketch the graph of $y = \frac{x^2+1}{x+1}$. Find all turning values, and describe the behaviour of the graph when x or y become large.

2. A pupil is swimming at the centre of a circular pond. At the edge of the pond there is a teacher, who wishes to catch the pupil, but who cannot swim. The teacher can run four times as fast as the pupil can swim, but not as fast as the pupil can run. Can the pupil escape from the teacher? Justify your answer.

3. Prove that, for every integer n:

   (a) $n^3 - n$ is divisible by 3;

   (b) $n^7 - n$ is divisible by 7;

   (c) $n^{13} - n$ is divisible by 13.

4. How many zeros are there at the end of the number 100! ? [*Note*: n! denotes the product of ali the integers from 1 to n inclusive.]

5. Prove that the product of four consecutive integers is always one less than a perfect square.

6. Let $(2+\sqrt{2})^n = P + F$, where P is an integer and $0 < F < 1$. Show that the positive integer n can be chosen so that $F > 0.999$.

7. Find the remainders upon dividing the polynomial $x + x^3 + x^9 + x^{27} + x^{81} + x^{243}$:

   (a) by $x - 1$;

   (b) by $x^2 - 1$.

8. Determine ali values of the coefficient a for which the equations $x^2 + ax + 1 = 0$ and $x^2 + x + a = 0$ have at least one common root.

9. If a, b, and c are any three positive real numbers, prove that

   $$(a+b)(b+c)(c+a) \geq 8abc.$$

   Show that equality holds only when $a = b = c$.

10. A chord of length $\sqrt{3}$ divides a circle of unit radius into two regions. Find the rectangle of maximum area which can be inscribed in the smaller of these two regions.