Blundon's inequality - proof and some corollaries

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Abstract

We consider a method for proving symmetric inequalities of three variables by making use of an inequality for the semi-perimeter, the inradius and the curcumradius of a triangle. This inequality was first proved by E. Rouche in 1851 but nowadays it is known in the literature as Blundon's (or Fundamental) inequality.

Contents

1 The inequality **1**

2 Corollaries **2**

3 Proving symmetric inequalities by means of Blundon's inequality **3**

1 The inequality

Theorem 1. (Blundon's inequality) For any triangle the following inequality holds true:

$$2R^2 + 10Rr - r^2 - 2(R-2r)\sqrt{R(R-2r)} \leq s^2 \leq 2R^2 + 10Rr - r^2 + 2(R-2r)\sqrt{R(R-2r)}.$$

Proof.

Let σ_1, σ_2 and σ_3 be the elementary symmetric functions of the sides a, b, c of a triangle. We first compute σ_1, σ_2 and σ_3 in terms of s,r and R. We have

$$\sigma_1 = a + b + c = 2s$$

and

$$\sigma_3 = abc = 4srR.$$

To compute σ_2 we use the Heron formula

$$A = \sqrt{s(s-a)(s-b)(s-c)},$$

where A is the area of the triangle with sidelengths a, b, c. Since $F = sr$ we have that

$$\begin{aligned} r^2 = \frac{(s-a)(s-b)(s-c)}{s} &= \frac{s^3 - s^2(a+b+c) + s(ab+bc+ca) - abc}{s} \\ &= -s^2 + \sigma_2 - 4Rr. \end{aligned}$$

Hence

$$\sigma_2 = s^2 + r^2 + 4Rr.$$

Consider the symmetric polynomial $(a-b)^2(b-c)^2(c-a)^2$ of a, b, c. We know that it can be written as a polynomial of σ_1, σ_2 and σ_3, and hence as a polynomial of s, r, R. More precisely, one checks easily that

$$\begin{aligned} (a-b)^2(b-c)^2(c-a)^2 &= \sigma_1^2\sigma_2^2 - 4\sigma_2^3 - 4\sigma_1^3\sigma_3 + 18\sigma_1\sigma_2\sigma_3 - 27\sigma_3^2 \\ &= -4r^2[(s^2 - 2R^2 - 10Rr + r^2)^2 - 4r(R-2r)^3]. \end{aligned}$$

Therefore $(s^2 - 2R^2 - 10Rs + r^2)^2 \leq 4R(R-2r)^3$ which is equivalent to the inequality.

□

Remark 1. The inequality is also sufficient for the existence of a triangle with semiperimeter s, inradius r and circumradius R. Moreover, Blundon has proved that it is the strongest possible inequality of the form $f(R, r) \leq s^2 \leq F(R, r)$, where $f(R, r)$ and $F(R, r)$ are homogeneous functions, with simultaneous equality only for equilateral triangles.

2 Corollaries

Corollary 1. *For any triangle the following inequalities hold true:*

$$(a)\ 27r^2 \leq \frac{27Rr}{2} \leq 16Rr - 5r^2 \leq s^2;$$

$$(b)\ s^2 \leq 4R^2 + 4Rr + 3r^2 \leq \frac{27}{4}R^2;$$

$$(c)\ 24Rr - 12r^2 \leq a^2 + b^2 + c^2 \leq 8R^2 + 4r^2;$$

$$(d)\ 6\sqrt{3}r \leq a + b + c \leq 4R + (6\sqrt{3} - 8)r.$$

Proof.

(a) It is easy to check that $\sqrt{R(R-2r)} \leq R-r$. Hence it follows form **Blundon's inequality** that

$$s^2 \leq 2R^2 + 10Rr - r^2 + 2(R-2r)\sqrt{R(R-2r)} \leq$$
$$\leq 2r^2 + 10Rr - r^2 - 2(R-2r)(R-r) = 4R^2 + 4Rr + 3r^2$$

The inequalities $16Rr - 5r^2 \geq \frac{27Rr}{2}$ and $\frac{27Rr}{2} \geq 27r^2$ are equivalent to $R \geq 2r$.

(b) It follows from **Blundon's inequality** that

$$s^2 \leq 2R^2 + 10Rr - r^2 + 2(R-2r)\sqrt{R(R-2r)}$$
$$\leq 2R^2 + 10Rr - r^2 + 2(R-2r)(R-r) = 4R^2 + 4Rr + 3r^2.$$

The inequality $4R^2 + 4Rr + 3r^2 \leq \frac{27}{4}R^2$ is equivalent to $(R-2r)(11R+6r) \geq 0$ which follows from **Euler's inequality**.

(c) We have that

$$a^2 + b^2 + c^2 = 4s^2 - 2(ab+bc+ca) = 2s^2 - 2r^2 - 8Rr$$

and the two inequalities follow from Euler's and Blundon's inequalities, respectively.

(d) It follows from Euler's and Blundon's inequalities that

$$a+b+c) = 4s^2 \geq 108r^2, \text{ i.e. } a+b+c \geq 6\sqrt{3}r.$$

The right hand side inequality follows easily since **Euler's inequality** implies that

$$\left[2R + (3\sqrt{3}-4)r\right]^2 \geq 4R^2 + 4Rr + 3r^2.$$

3 Proving symmetric inequalities by means of Blundon's inequality

Let a, b, c be the sidelengths of a triangle. Set

$$(1) \qquad x = s-a, y = s-b, z = s-c.$$

Then the triangle inequality implies that x, y and z are positive numbers. Conversely, if x, y and z are positive numbers, then $a = x+y$, $b = y+z$, $c = z+x$

are sidelengths of a triangle satisfying (1). Denote by σ_1, σ_2 and σ_3 the elementary symmetric functions of x, y and z. Then we obtained that

$$\sigma_1 = (s-a)+(s-b)+(s-c) = s,$$

$$\sigma_2 = (s-a)(s-b)+(s-b)(s-c)+(s-c)(s-a) = 3s^2-8s(a+b+c)+ab+bc+ca = 4Rr+r^2$$

and

$$\sigma_3 = (s-a)(s-b)(s-c) = sr^2$$

Or in general

$$\text{(2)} \qquad \sigma_1 = s,\ \sigma_2 = 4Rr + r^2,\ \sigma_3 = sr^2.$$

Suppose now that $f(x, y, z)$ is a symmetric polynomial and we have to prove that $f(x, y, z) \geq 0$ for all positive numbers x, y, z. Then we may proceed on the following way. First, using the substitution (1) and formulas (2) we write the inequality $f(x, y, z) \geq 0$ as of s, r and R and then prove it by meaning use of **Blundon's inequality** or some of its consequences listed above.

If we have to prove an inequality $f(x, y, z) \geq 0$ for all positive numbers x, y, z with the condition $x + y + z = 1$ then we can use the substitutions

$$x = \frac{s-a}{s}, y = \frac{s-b}{s}, z = \frac{s-c}{s}.$$

In this case

$$\sigma_1 = 1, \sigma_2 = \frac{4Rr + r^2}{s^2}, \sigma_3 = \frac{r^2}{s^2}$$

and we proceed as above. □