Arithmetic Compensation Method

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The Arithmetic Compensation Method is a powerful tool which can be used to prove certain difficult symmetric inequalities. Such a problem, which was left unsolved on the Mathlnks Inequalities Forum, is presented here:

Problem. *Let $a, b, c, d \geq 0$ such that $a+b+c+d=4$. For $p > \frac{64}{27}$, what is the minimum value of the expression*

$$\frac{1}{p-abc}+\frac{1}{p-bcd}+\frac{1}{p-cda}+\frac{1}{p-dab}?$$

Arithmetic Compensation Theorem (Short Form). *Let $s > 0$ and let $F(x_1, x_2, \cdots, x_n)$ be a symmetrical continuous function on the compact set in $\mathbb{R}^n$*

$$S = \{(x_1, x_2, \ldots, x_n)\colon x_1 + x_2 + \cdots + x_n = s,\ x_1 \geq 0, \ldots, x_n \geq 0\}.$$

If

$$F(x_1, x_2, x_3, \ldots, x_n) < \max\{F(\frac{x_1+x_2}{2}, \frac{x_1+x_2}{2}, x_3, \ldots, x_n), F(0, x_1+x_2, x_3, \ldots, x_n)\}$$

for all $(x_1, x_2, \ldots, x_n) \in S$ with $x_1 > x_2 > 0$, then

$$F(x_1, x_2, \ldots, x_n) \leq \max_{1\leq k\leq n} F(\frac{s}{k}, \ldots, \frac{s}{k}, 0, \ldots, 0)$$

for all $(x_1, x_2, \ldots, x_n) \in S$.

Proof. Since the function F is continuous on the compact set S, F attains a maximum value at one or more points of the set. Let $(x_1, x_2, \ldots, x_n)$ be such a maximum point. For the sake of contradiction, assume that there exist two numbers x_i and x_j such that $x_i > x_j > 0$; for convenience, let us consider $i = 1$ and $j = 2$ (hence $x_1 > x_2 > 0$). According to the hypothesis, we have

$$F(x_1, x_2, x_3, \ldots, x_n) < \max\{F(\frac{x_1+x_2}{2}, \frac{x_1+x_2}{2}, x_3, \ldots, x_n), F(0, x_1+x_2, x_3, \ldots, x_n)\}.$$

But this is false because F is maximal at $(x_1, x_2, \ldots, x_n)$, and the theorem is proved. □

Arithmetic Compensation Theorem (Extended Form). *Let $s > 0$ and let $F(x_1, x_2, \cdots, x_n)$ be a symmetrical continuous function on the compact set in $\mathbb{R}^n$*

$$S = \{(x_1, x_2, \ldots, x_n)\colon\ x_1 + x_2 + \cdots + x_n = s,\ x_1 \geq 0, \ldots, x_n \geq 0\}.$$

If

$$F(x_1, x_2, x_3, \ldots, x_n) \leq \max\{F(\frac{x_1 + x_2}{2}, \frac{x_1 + x_2}{2}, x_3, \ldots, x_n), F(0, x_1 + x_2, x_3, \ldots, x_n)\}$$

for all $(x_1, x_2, \ldots, x_n) \in S$ with $x_1 > x_2 > 0$, then

$$F(x_1, x_2, \ldots, x_n) \leq \max_{1 \leq k \leq n} F(\frac{s}{k}, \ldots, \frac{s}{k}, 0, \ldots, 0)$$

for all $(x_1, x_2, \ldots, x_n) \in S$.

Proof. In order to prove this theorem, we will show that among the maximum points of F there exists at least one point $(y_1, y_2, \ldots, y_n)$ such that all $y_i \in \{0, \frac{s}{k}\}$, where $1 \leq k \leq n$. Let $(x_1, x_2, \ldots, x_n)$ be a maximum point. Again suppose by way of contradiction that $x_1 > x_2 > 0$. We have considered the case where the inequality in the hypothesis is strict; we now prove the conclusion for the equality case in the hypothesis; that is, when

$$F(x_1, x_2, x_3, \ldots, x_n) = \max\{F(\frac{x_1 + x_2}{2}, \frac{x_1 + x_2}{2}, x_3, \ldots, x_n), F(0, x_1 + x_2, x_3, \ldots, x_n)\}$$

The function F attains again its maximum value at $(y_1, y_2, \ldots, y_n)$ with $y_i = x_i$ for $i \geq 3$ and either $y_1 = y_2 = \frac{x_1 + x_2}{2}$ or $y_1 = 0$ and $y_2 = x_1 + x_2$. If there are not two numbers y_i and y_j such that $y_i > y_j > 0$, then the proof is finished. Otherwise, we iterate the preceding process, eventually in the limiting case finding a maximum point $(z_1, z_2, \ldots, z_n)$ such that all $z_i \in \{0, \frac{s}{k}\}$, where $1 \leq k \leq n$. $\square$

Applications

Problem 1. *If $a, b, c, d \geq 0$ such that $a + b + c + d = 4$, then*

$$\frac{1}{5 - abc} + \frac{1}{5 - bcd} + \frac{1}{5 - cda} + \frac{1}{5 - dab} \leq 1.$$

Solution. If at least two of the numbers a, b, c, d are equal to zero, then the inequality is clearly true. Otherwise, let us denote by $F(a, b, c, d)$ the left hand side of the inequality. We claim that for $a > b > 0$, the inequality of the theorem holds; that is, we claim that

$$F(a, b, c, d) < \max\{F(\frac{a + b}{2}, \frac{a + b}{2}, c, d), F(0, a + b, c, d)\} \quad (1)$$

Then, by the short form of the theorem, it follows that

$$F(a,b,c,d) < \max\{F(4,0,0,0), F(2,2,0,0), F(\frac{4}{3},\frac{4}{3},\frac{4}{3},0), F(1,1,1,1)\}.$$

Since $F(4,0,0,0) = F(2,2,0,0) = \frac{4}{5}$, $F(\frac{4}{3},\frac{4}{3},\frac{4}{3},0) = \frac{348}{355}$, and $F(1,1,1,1) = 1$, we see that $F(a,b,c,d) \le 1$, which is the desired inequality.

In order to prove (1), we will assume for the sake of contradiction that there exist $a > b > 0$, $c > 0$ and $d \ge 0$ such that

$$\begin{aligned} F(a,b,c,d) &\ge F(t,t,c,d) \text{ and} \\ F(a,b,c,d) &\ge F(0,2t,c,d), \end{aligned}$$

where $t = \frac{a+b}{2}$. Write now the inequality $F(a,b,c,d) \ge F(t,t,c,d)$ in the form

$$\frac{2(5-tcd)}{(5-acd)(5-bcd)} - \frac{2}{5-tcd} \ge (\frac{1}{5-t^2c} - \frac{1}{5-abc}) + (\frac{1}{5-t^2d} - \frac{1}{5-abd}).$$

Dividing by the positive factor $t^2 - ab$, the inequality becomes

$$\frac{2c^2d^2}{(5-acd)(5-bcd)(5-tcd)} \ge \frac{c}{(5-acd)(5-t^2c)} + \frac{d}{(5-abd)(5-t^2d)}.$$

Since

$$\frac{c}{(5-acd)(5-t^2c)} + \frac{d}{(5-abd)(5-t^2d)} > \frac{c}{5(5-t^2c)} + \frac{d}{5(5-t^2d)},$$

we get

$$\frac{2c^2d^2}{(5-acd)(5-bcd)(5-tcd)} > \frac{c}{5(5-t^2c)} + \frac{d}{5(5-t^2d)}. \tag{2}$$

Similarly, write the inequality $F(a,b,c,d) \ge F(0,2t,c,d)$ as follows:

$$(\frac{1}{5-abc} - \frac{1}{5}) + (\frac{1}{5-abd} - \frac{1}{5}) + (\frac{1}{5-acd} - \frac{1}{5-bcd}) \ge \frac{1}{5} + \frac{1}{5-2tcd}$$

$$\frac{abc}{5(5-abc)} + \frac{abd}{5(5-abd)} + \frac{2(5-tcd)}{(5-acd)(5-bcd)} \ge \frac{2(5-tcd)}{5(5-2tcd)}$$

$$\frac{c}{5(5-abc)} + \frac{d}{5(5-abd)} \ge \frac{2c^2d^2(5-tcd)}{5(5-acd)(5-bcd)(5-2tcd)}.$$

Since

$$\frac{5-tcd}{5-2tcd} \ge \frac{5}{5-tcd},$$

we get

$$\frac{c}{5(5-abc)} + \frac{d}{5(5-abd)} \ge \frac{2c^2d^2}{(5-acd)(5-bcd)(5-tcd)},$$

which contradicts (2). This completes the proof. Equality occurs if and only if $a = b = c = d = 1$.

Problem 2. *If $a, b, c, d \geq 0$ such that $a+b+c+d=4$, then*

$$\frac{1}{4-abc}+\frac{1}{4-bcd}+\frac{1}{4-cda}+\frac{1}{4-dab} \leq \frac{15}{11}.$$

Solution. If at least two of the numbers a, b, c, d are equal to zero, then the inequality is true. Otherwise, as in the preceding problem, we can show that

$$F(a,b,c,d) < \max\{F(\frac{a+b}{2}, \frac{a+b}{2}, c, d), F(0, a+b, c, d)\}.$$

By the theorem, we have

$$F(a,b,c,d) < \max\{F(4,0,0,0), F(2,2,0,0), F(\frac{4}{3}, \frac{4}{3}, \frac{4}{3}, 0), F(1,1,1,1)\}.$$

Since $F(4,0,0,0) = F(2,2,0,0) = 1$, $F(\frac{4}{3}, \frac{4}{3}, \frac{4}{3}, 0) = \frac{15}{11}$, and $F(1,1,1,1) = \frac{4}{3}$, the conclusion follows. Equality occurs when one of a, b, c, d equals zero and the other three equal $\frac{4}{3}$.

Problem 3. *If $a, b, c, d \geq 0$ such that $a+b+c+d=1$, then*

$$\frac{(1+2a)(1+2b)(1+2c)(1+2d)}{(1-a)(1-b)(1-c)(1-d)} \geq \frac{125}{8}.$$

Solution. Let

$$F(a,b,c,d) = -\frac{(1+2a)(1+2b)(1+2c)(1+2d)}{(1-a)(1-b)(1-c)(1-d)}.$$

We claim that for $a > b > 0$,

$$F(a,b,c,d) < \max\{F(\frac{a+b}{2}, \frac{a+b}{2}, c, d), F(0, a+b, c, d)\}. \tag{3}$$

Then, by the extended form of the theorem, we have

$$F(a,b,c,d) \leq \max\{F(\frac{1}{2}, \frac{1}{2}, 0, 0), F(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, 0), F(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4})\}.$$

Since $F(\frac{1}{2}, \frac{1}{2}, 0, 0) = -16$, $F(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, 0) = -\frac{125}{8}$, and $F(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}) = -16$, we get $F(a,b,c,d) \leq -\frac{125}{8}$ which is the desired inequality.

The inequality (3) is equivalent to

$$\frac{(1+2a)(1+2b)}{(1-a)(1-b)} \geq \min\{(\frac{1+2t}{1-t})^2, \frac{1+4t}{1-2t}\}.$$

The inequality

$$\frac{(1+2a)(1+2b)}{(1-a)(1-b)} \geq (\frac{1+2t}{1-t})^2$$

is equivalent to

$$\frac{3(4t-1)(t^2-ab)}{(1-t)(1-a)(1-b)} \geq 0, \tag{4}$$

and the inequality

$$\frac{(1+2a)(1+2b)}{(1-a)(1-b)} \geq \frac{1+4t}{1-2t}$$

is equivalent to

$$\frac{3ab(-4t+1)}{(1-2t)(1-a)(1-b)} \geq 0. \tag{5}$$

Since (4) is true for $t \geq \frac{1}{4}$ and (5) is true for $t \leq \frac{1}{4}$, the proof is completed. Equality occurs when one of the numbers a, b, c, d is equal to zero, and the others equal $\frac{1}{3}$.