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$\boxed{1}$ Let H be the orthocenter of an acute-angled triangle ABC. The circle Γ_A centered at the midpoint of BC and passing through H intersects the sideline BC at points A_1 and A_2. Similarly, define the points B_1, B_2, C_1 and C_2.

Prove that six points A_1 , A_2, B_1, B_2, C_1 and C_2 are concyclic.

$\boxed{2}$ **(i)** If x, y and z are three real numbers, all different from 1, such that $xyz = 1$, then prove that $\frac{x^2}{(x-1)^2} + \frac{y^2}{(y-1)^2} + \frac{z^2}{(z-1)^2} \geq 1$. (With the $\sum$ sign for cyclic summation, this inequality could be rewritten as $\sum \frac{x^2}{(x-1)^2} \geq 1$.)

(ii) Prove that equality is achieved for infinitely many triples of rational numbers x, y and z.

$\boxed{3}$ Prove that there are infinitely many positive integers n such that $n^2 + 1$ has a prime divisor greater than $2n + \sqrt{2n}$.