Questions for the 2019 pde work

1. Introduction

First read the text p.1-9, p.12-17 and try to understand it the best as you can. When you answer questions, explain what you are doing and for a proof, give all the steps.

2. Maths Questions

2.1

Recall the theorem that gives the general solution of a quasilinear partial differential equation of the first order in two dimensions x, y.
Why the two first integral must be functionally independent and not only linear independent? (don't give a proof, just an explanation).

2.2

Find the general solution of the p.d.e:

$$\frac{\partial \rho}{\partial t} + v(\rho)\frac{\partial \rho}{\partial x} = 0 \tag{1}$$

where $\rho(x,t)$ is a C^1 function and using the notations of the text.

2.3

Find in an explicit way the solution of $\begin{cases} \dfrac{\partial \rho}{\partial t} + v(\rho)\dfrac{\partial \rho}{\partial x} = 0 \\ \rho(x,0) = f(x) \end{cases}$

where $\rho(x,t)$, $f(x)$ are C^1 functions. (Hint: use the implicit function theorem)

2.4

Prove that the previous solution $\rho(x,t)$ exists for t as long as

$$1 + tv'(\rho)f'(x - v(\rho)t) > 0$$

Hint: use the implicit theorem with 3 variables

2.5

Prove that:

$$\frac{\partial \rho}{\partial x} = \frac{f'(x - v(\rho)t)}{1 + tv'(\rho)f'(x - v(\rho)t)}$$

$$\frac{\partial \rho}{\partial t} = -\frac{v(\rho)f'(x - v(\rho)t)}{1 + tv'(\rho)f'(x - v(\rho)t)}$$

2.6

What happens with $\frac{\partial \rho}{\partial x}$ and $\frac{\partial \rho}{\partial t}$ when $1 + tv'(\rho)f'(x - v(\rho)t)$ tends to zero? (Comment: the solution develops a discontinuity known as a shock)

2.7

Prove that all the characteristics curves of (1) are straight lines

2.8. *Example*

2.8.1

Solve explicitly: $\begin{cases} \dfrac{\partial \rho}{\partial t} + \rho \dfrac{\partial \rho}{\partial x} = 0 \\ \rho(x, 0) = -x \end{cases}$

2.8.2

For which t is the solution $\rho(x, t)$ defined? Check this by two way: with the computation of 2.8.1 or with the condition you proved in 2.4

2.8.3

Prove that all the characteristic lines intersect at the point $(0, 1)$ in the (x, t) plane. What happens there? Why the solutions cannot exist at this point?