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Algebraic solution of Fermat's theorem
(mathematics, number theory)

Abstract: *Fermat's Last Theorem (or Fermat's last theorem) is one of the most popular theorems in mathematics. Formulated in French mathematician Pierre Fermat in 1637. Despite the simplicity of the formulation, literally, at the “school” arithmetic level, proof of the theorem sought by many mathematicians for more than three hundred years. And only in 1994 year the theorem was proven by the English mathematician Andrew Wilson with colleagues; The proof was published in 1995. [1]-[5]*

The author of this article has been searching for his own for a long time. accessible algebraic solution to this problem and believes that he succeeded, which he presents in this article.

Keywords: *Theorem, Fermat, elementary, solution*

Introduction.

$$X^n + Y^n = Z^n \quad (01)$$

where:

n - prime number, $n > 2$; X, Y, Z are integers.

The solutions of which can be X, Y, Z - relatively prime numbers.

1. Decomposition of (01) into multipliers.

If n is odd, then (01) will decompose into multipliers:

$$X^n + Y^n = (X + Y)(X^{n-1} - X^{n-2}Y + \dots - XY^{n-2} + Y^{n-1}) \quad (02)$$

where in the second bracket is the geometric progression

first term $a_1 = X^{n-1}$, and a multiplier $q = -\frac{Y}{X}$

The sum of the members of which $S = \frac{a_1(1 - q^n)}{1 - q}$

$$Z^n = Z_{11} Z_{22} \quad (03)$$

where:

$$Z_{11} = X + Y \quad (04)$$

$$Z_{22} = X^{n-1} - X^{n-2}Y + \dots - X Y^{n-1} + Y^{n-1} \quad (05)$$

2. Equivalent representation Z_{22} .

If we sum the equidistant terms from the middle term of the progression Z_{22} in pairs.

of the middle term of the progression in pairs we have:

for degree 3

$$Z_{22} = (X + Y)^2 - 3XY \quad (06)$$

Fifth degree :

$$Z_{22} = \frac{X^5 + Y^5}{X + Y} = X^4 - X^3 Y + X^2 Y^2 - X Y^3 + Y^4 \quad (07)$$

$$X^4 + Y^4 = (X + Y)^4 - 4 X Y (X + Y)^2 + 2 X^2 Y^2 \quad (08)$$

$$-X Y^3 - X^3 Y = -X Y (X^2 + Y^2) = -X Y (X + Y)^2 + 2 X^2 Y^2 \quad (09)$$

$$Z_{225} = (X + Y)^4 - 5(X + Y)^2 + 5 X^2 Y^2 \quad (10)$$

to the 7th degree:

$$Z_{227} = (X + Y)^6 - 7 X Y (X + Y)^4 + 14 X^2 Y^2 (X + Y)^2 - 7 X^3 Y^3 \quad (11)$$

degree n :

$$\begin{aligned} Z_{22N} &= \frac{X^n + Y^n}{X + Y} = \\ &= (X + Y)^{n-1} - K_{n-3}XY(X + Y)^{n-3} + \dots \mp K_2 X^{\frac{n-3}{2}} Y^{\frac{n-3}{2}} (X + Y)^2 \pm n X^{\frac{n-1}{2}} Y^{\frac{n-1}{2}} \end{aligned} \quad (12)$$

$$Z_{22N} = (X + Y)^{n-1} - K_{n-3}XY(X + Y)^{n-3} + \dots \mp K_2 X^{\frac{n-3}{2}} Y^{\frac{n-3}{2}} (X + Y)^2 \pm n X^{\frac{n-1}{2}} Y^{\frac{n-1}{2}} \quad (*)$$

where $K_{n-3} \dots K_2$ corresponding coefficients at $(XY) \dots (X + Y) \dots$

equivalent representation Z_{22N} algebraic sum of even powers of

$X + Y$ and the residual term $\pm n X^{\frac{n-1}{2}} Y^{\frac{n-1}{2}}$.

Lemma1: Suppose that for an n odd number and for the previous $n-2$

an equivalent representation() is valid, then for the next $n+2$ it is*

() is valid.*

We show the transition from the two previous odd degrees to the next one

further:

$$Z_n^n = X^n + Y^n \quad (14a)$$

$$(X^{n-2} + Y^{n-2})(X^2 + Y^2) = X^n + Y^n + Y^2 X^{n-2} + Y^2 Y^{n-2} \quad (14b)$$

$$Z_n^n = X^n + Y^n = (X^{n-2} + Y^{n-2})(X^2 + Y^2) - X^2 Y^2 (X^{n-4} + Y^{n-4}) \quad (14c)$$

details:

$$\frac{X^{n_1} + Y^{n_1}}{X + Y} \quad \text{multiply by} \quad (X + Y)^2 - 2 X Y$$

$$X^{n+2} + Y^{n+2} = (X + Y)[Z_{22N}(X + Y)^2 - 2XYZ_{22N} - X^2 Y^2 (X^{n-2} + Y^{n-2})] \quad (14)$$

$$(X + Y)^{n-1} - K_{n-3} XY (X + Y)^{n-3} + \dots \mp K_2 X^{\frac{n-3}{2}} Y^{\frac{n-3}{2}} (X + Y)^2 \pm n X^{\frac{n-1}{2}} Y^{\frac{n-1}{2}} *$$

$$* (X + Y)^2 = (X + Y)^{n+1} - K_{n-1(01)} XY (X + Y)^{n-1} + \dots \mp K_{4(01)} X^{\frac{n-3}{2}} Y^{\frac{n-3}{2}} \pm n X^{\frac{n-1}{2}} Y^{\frac{n-1}{2}} (X + Y)^2 \quad (15)$$

$$-2 XY [(X + Y)^{n-1} - K_{n-3} XY (X + Y)^{n-3} + \dots \mp K_2 X^{\frac{n-3}{2}} Y^{\frac{n-3}{2}} (X + Y)^2 \pm n X^{\frac{n-1}{2}} Y^{\frac{n-1}{2}}] =$$

=

$$-2 XY (X + Y)^{n-1} - K_{n-3(02)} 2 X^2 Y^2 (X + Y)^{n-3} + \dots \mp K_{2(02)} 2 X^{\frac{n-1}{2}} Y^{\frac{n-1}{2}} (X + Y)^2 \pm 2 X^{\frac{n+1}{2}} Y^{\frac{n+1}{2}} \quad n$$

(16)

$$-X^2 Y^2 (X + Y)^{n-3} + K_{n-3(03)} X^3 Y^3 (X + Y)^{n-5} + \dots \pm K_{2(03)} X^{\frac{n-1}{2}} Y^{\frac{n-1}{2}} (X + Y)^2 \mp (n-2) X^{\frac{n+1}{2}} Y^{\frac{n+1}{2}}$$

(17)

where $K_{...(01)}$ - corresponding coefficients when multiplied by $(X + Y)^2$,

$K_{...(02)}$ - corresponding coefficients when multiplied by $-2XY$,

$K_{...(03)}$ - corresponding coefficients when multiplied by $-X^2 Y^2$

After adding these algebraic terms we again obtain(*)

Theorem1.

The equivalent representation () is valid for any prime n.*

By lemma , if the two previous representations of () are valid,*

of degree 3 and 5, then it is valid for degree 7. Now

taking the previous 5 and 7 degrees we have its validity for the 9th degree, etc,

which means all odd degrees are described by the above formula.

And since it includes prime n , it is valid for prime n.

Let us represent (1) as:

$$\begin{aligned}
 (X+Y)^n - Z^n &= nX^{n-1}Y + \frac{n(n-1)}{2}X^{n-2}Y^2 + \dots + \frac{n(n-1)}{2}Y^{n-2}X^2 + nXY^{n-1} \\
 (X+Y-Z)[(X+Y-Z)^{n-1} - nk_{n-3}(X+Y)Z(X+Y-Z)^{n-3} \pm nk_2(X+Y)^{n-3}Z^{n-3}(X+Y-Z)^2 \mp n(XY)^{\frac{n-1}{2}}] \\
 &= nX^{n-1}Y + \frac{n(n-1)}{2}X^{n-2}Y^2 + \dots + \frac{n(n-1)}{2}Y^{n-2}X^2 + nXY^{n-1}
 \end{aligned} \tag{18}$$

it follows:

$$Z_{22} = (X+Y)^{n-1} - nXY(\dots) \tag{19}$$

What indicates the presence of n in $X+Y-Z$.

And let's separate the common multiplier n :

$$Z_{22n} = (X+Y)^{n-1} - nk_{n-3}XY(X+Y)^{n-3} + \dots \pm nk_2X^{\frac{n-3}{2}}Y^{\frac{n-3}{2}}(X+Y)^2 \mp nX^{\frac{n-1}{2}}Y^{\frac{n-1}{2}} \tag{20}$$

3. Analysis of Equation (20).

From equation (20) $Z_{11} = X+Y$ and Z_{22} cannot have a common factor for except for n . From which the following equalities follow in the absence of n :

$$X+Y=Z_1^n, \quad Z-X=Y_1^n, \quad Z-Y=X_1^n \tag{21}$$

$$Z_{11}=Z_1^n, \quad Z_{22}=Z_2^n, \quad X_{11}=X_1^n, \quad X_{22}=X_2^n, \quad Y_{11}=Y_1^n, \quad Y_{22}=Y_2^n \tag{22}$$

$$X+Y-Z=nX_1Y_1Z_1K_o \tag{23}$$

where

K_o -an integer coprime to the others specified

except n .

$$Z_1^n = X_1^n + Y_1^n + 2nX_1Y_1Z_1K_o \tag{24}$$

$$X-Y = X_1^n - Y_1^n \tag{25}$$

$$Z_1^n - Z = n X_1 Y_1 Z_1 K_o \quad (26)$$

$$Z_2 = Z_1^{n-1} - n X_1 Y_1 K_o \quad (27)$$

$$X - X_1^n = n X_1 Y_1 Z_1 K_o \quad (28)$$

$$X_2 = X_1^{n-1} + n Z_1 Y_1 K_o \quad (29)$$

$$Y - Y_1^n = n X_1 Y_1 Z_1 K_o \quad (30)$$

$$Y_2 = Y_1^{n-1} + n Z_1 X_1 K_o \quad (31)$$

$$2 X = Z_1^n - Y_1^n + X_1^n \quad (32)$$

$$2 Y = Z_1^n - X_1^n + Y_1^n \quad (33)$$

$$2 Z = Z_1^n + X_1^n + Y_1^n \quad (34)$$

$$Z_1^n - X_1^n - Y_1^n = 2 n X_1 Y_1 Z_1 K_o \quad (35)$$

$$Z_1^n - [(X_1 + Y_1)^n - n X_1^{n-1} Y_1 - \dots - n Y_1^{n-1} X_1] = 2 n X_1 Y_1 Z_1 K_o \quad (36)$$

$$Z_1 - X_1 - Y_1 = n K_n \text{ from which it follows } Z_1 > n \quad (37)$$

If the sum or difference of two coprime numbers has a factor n , then

the sum and difference of the n -power of these numbers is divisible by at least n^2 ,

which is obvious from (20), (04).

If in the expansion Z, X, Y has a prime factor n

$$Z_{22} = nZ_2^n, \quad X_{22} = nX_2^n, \quad Y_{22} = nY_2^n \quad (38)$$

and according to formula (20) Z_2 cannot have n available, otherwise

this will lead to the presence of it in X or Y , and vice versa, which is not acceptable.

Z_2, X_2, Y_2 - does not contain the factor n . In this regard, if Z contains a factor n , then

formula (26) has the form, since sum $X_1^n + Y_1^n$ contains a multiplier n^m where

natural number, $m \geq 2$ and

Z_2, X_2, Y_2 - does not contain the factor n .

$$n^{nm-1} Z_1^n = X_1^n + Y_1^n + 2n^m X_1 Y_1 Z_1 K_o \quad (39)$$

To solve (34) in integers, degree n in $X_1^n + Y_1^n$, should be equal degree n in the last monomial, that is, minimally n^2 .

similar:

$$n^{nm-1} X_1^n = Z_1^n - Y_1^n - 2n^m X_1 Y_1 Z_1 K \quad (40)$$

$$n^{nm-1} Y_1^n = Z_1^n - X_1^n - 2n^m X_1 Y_1 Z_1 K \quad (41)$$

$$n^{nm-1} Z_1^n - n^m Z_1 Z_2 = n^m X_1 Y_1 Z_1 K, \quad Z = n^m Z_1 Z_2 \quad (42)$$

$$n^m X_1 X_2 - n^{nm-1} X_1^n = n^m X_1 Y_1 Z_1 K, \quad X = n^m X_1 X_2 \quad (43)$$

$$n^m Y_1 Y_2 - n^{nm-1} Y_1^n = n^m X_1 Y_1 Z_1 K, \quad Y = n^m Y_1 Y_2 \quad (44)$$

What follows:

$$Z_2 = n^{nm-m-1} Z_1^{n-1} - X_1 Y_1 K \quad (45)$$

$$X_2 = n^{nm-m-1} X_1^{n-1} + Z_1 Y_1 K \quad (46)$$

$$Y_2 = n^{nm-m-1} Y_1^{n-1} + Z_1 X_1 K \quad (47)$$

Thus

$$X + Y - Z = n X_1 Y_1 Z_1 K_o \text{ universal,}$$

where X_1, Y_1, Z_1, K_o -coprime corresponds to X, Y, Z with and without n . The difference is

$$K_o = n^{m-1} K \quad (48)$$

4. Solution of Fermat's theorem for odd n .

Using (32)-(34) we have:

$$(Z_1^n + X_1^n - Y_1^n)^n + (Z_1^n + Y_1^n - X_1^n)^n = (Z_1^n + X_1^n + Y_1^n)^n \quad (49)$$

see point 4 (25)...:

$$Z_1^n + 2n Z_1^{n(n-2)}(X-Y)^2 + \dots + 2 Z_1^n n (X-Y)^{n-1} = n Z_1^{n(n-1)}(2Z-X-Y) + \dots +$$

-3-

$$+ n Z_1^n (2Z-X-Y)^{n-1} + (2Z-X-Y)^n \quad (50)$$

--2-- --1--

We reduce Z_1^n to:

$$-1- \quad \frac{2^n Z^n}{Z_1^n} = 2^n Z_2^n \quad (51)$$

$$-2- \quad 2^{n-1} Z^{n-1} = 2^{n-1} Z_1^{n-1} Z_2^{n-1} \quad (52)$$

$$-3- \quad (X-Y)^{n-1} = (X+Y-2Y)^{n-1} = Z_1^n - (n-1)Z_1^{n(n-1)} 2Y + \dots + 2^{n-1} Y^{n-1} \quad (53)$$

In (20), we transform the last monomial:

$$\pm n X^{\frac{n-1}{2}} Y^{\frac{n-1}{2}} = \pm n Y^{\frac{n-1}{2}} (Z_1^n - Y)^{\frac{n-1}{2}} = n Z_1^{n \frac{n-1}{2}} + \dots + n Y^{n-1} \quad (54)$$

Thus, the difference with -3-(53) is

$2^n Z_2^n - 2^n Y^{n-1}$ and as a result, the sum of monomials with a degree not lower than Z_1^n and only one monomial -2- with degree Z_1^{n-1} . Take Z_1 from neighbors monomial -2- is not possible. If you assign a factor n , then it is available in all other monomials.

Similarly regarding and Y_1 , X_1 , and consider the case with the presence there is no need for a factor n in X or Y or Z . Therefore, in whole numbers there is no solution! Impossible to have $Z_{22} = Z_2^n$, $X_{22} = X_2^n$, $Y_{22} = Y_2^n$, but only inequalities.

Q.E.D!

5. Degree $n=3$.

According to (28) and Newton's binomial[6]:

$$\begin{aligned} Z_2^3 &= Z_1^6 - 3(X_1^3 + 3 X_1 Y_1 Z_1 K_o)(Y_1^3 + 3 X_1 Y_1 Z_1 K_o) = (Z_1^2 - 3 X_1 Y_1 K_o)^3 = \\ &= Z_1^6 - 9 Z_1^4 X_1 Y_1 K_o + 27 Z_1^2 X_1^2 Y_1^2 K_o^2 - 27 X_1^3 Y_1^3 K_o^3 \end{aligned} \quad (55)$$

On the other side :

$$\begin{aligned} Z_2^3 &= (X + Y)^3 - 3 X Y = \\ &= Z_1^6 - 3 X_1^3 Y_1^3 - 9 X_1^3 X_1 Y_1 Z_1 K_o - 9 Y_1^3 X_1 Y_1 Z_1 K_o - 27 X_1^2 Y_1^2 Z_1^2 K_o^2 \end{aligned} \quad (56)$$

In (43), the underlined terms are divided into Z_1^2 (36), (37) and $K_o=1$. What follows:

$$\frac{8 X_1^3 Y_1^3}{Z_1^2} = \text{integer} \quad (57)$$

(57)-impossible due to (37).

And there is no need to consider cases where n is in X or Y or Z ,

since it is not available in the two versions that we use.

5. Conclusion.

If the degree in (01) is odd, there is no solution. Pharm proved the absence of a solution

for the 4th degree and thereby proved its absence for everyone $n=2^m$,

where m is an integer.

Fermat's theorem is solvable in the first and second powers!

Literature.

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