PROOF OF BEAL'S CONJECTURE

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Beal's Conjecture

If $A^x + B^y = C^z$, $x > 2$, $y > 2$, $z > 2$, where A, B, C, x, y and z are positive integers, then A, B and C have a highest common factor>1.

Beal's conjecture equation can be written as $C^z - B^y = A^x$. Let it be initially assumed that C and B have a highest common factor of 1 i.e. they are coprime. If $n > 2$ is an integer, algebraic factorisation of the left hand side of the equation gives,

$C^z - B^y = (C^{z/n})^n - (B^{y/n})^n$

$= [C^{z/n} - B^{y/n}][(C^{z/n})^{n-1} + (C^{z/n})^{n-2} B^{y/n} + (C^{z/n})^{n-3} (B^{y/n})^2 +... + (C^{z/n})^2 (B^{y/n})^{n-3} + C^{z/n}(B^{y/n})^{n-2} + (B^{y/n})^{n-1}]$

For $C^z - B^y = A^x$ i.e.

$[C^{z/n} - B^{y/n}][(C^{z/n})^{n-1} + (C^{z/n})^{n-2} B^{y/n} + (C^{z/n})^{n-3} (B^{y/n})^2 +... + (C^{z/n})^2 (B^{y/n})^{n-3} + C^{z/n}(B^{y/n})^{n-2} + (B^{y/n})^{n-1}] = A^x$, then every term in each of the 2 brackets must be integers and not irrational numbers. This therefore implies that $C^{z/n}$ and $B^{y/n}$ must be integers. Note that if C and B have a highest common factor of 1 i.e. they are coprime, then $C^{z/n}$ and $B^{y/n}$ have a highest common factor of 1 i.e. they are coprime. Let $C^{z/n} = R$ and $B^{y/n} = Q$. Hence we can re-write $C^z - B^y = (C^{z/n})^n - (B^{y/n})^n$

$= R^n – Q^n$

$= (R – Q)(R^{n-1} + R^{n-2} Q + R^{n-3} Q^2 +... + R^2 Q^{n-3} + RQ^{n-2} + Q^{n-1})$

Fermat's last theorem says that $P^n + Q^n \neq R^n$, $n > 2$ if P, Q, R and n are positive integers. It is sufficient to let P, Q and R have a highest common factor of 1, i.e. they are coprime, because if they had any highest common factor >1, this can be factored out and cancelled out of the Fermat inequality. The Fermat inequality can be re-written as

$R^n – Q^n \neq P^n$

i.e. $(R – Q)(R^{n-1} + R^{n-2} Q + R^{n-3} Q^2 +... + R^2 Q^{n-3} + RQ^{n-2} + Q^{n-1}) \neq P^n$

Beal's conjecture of the form $C^z - B^y = A^x$ is therefore proved by using Fermat's Last Theorem. If R and Q on the left hand side of the inequality have a highest common factor $= k >1$, this can be factored out to give

$(R – Q)(R^{n-1} + R^{n-2} Q + R^{n-3} Q^2 +... + R^2 Q^{n-3} + RQ^{n-2} + Q^{n-1})$

$= k^n(U – T)(U^{n-1} + U^{n-2} T + U^{n-3} T^2 +... + U^2 T^{n-3} + UT^{n-2} + T^{n-1})$

where $R = kU$ and $Q = kT$.
The introduction of the extra algebraic term, k^n, is sufficient to allow
$k^n(U - T)(U^{n-1} + U^{n-2} T + U^{n-3} T^2 +... + U^2 T^{n-3} + UT^{n-2} + T^{n-1}) = A^x$
where $n \neq x$.

Consider Beal's conjecture written in its usual form $A^x + B^y = C^z$
Let it be initially assumed that A and B have a highest common factor of 1 i.e. they are coprime. If $n > 2$ is an odd integer, algebraic factorisation of the left hand side of the equation gives,
$A^x + B^y = (A^{x/n})^n + (B^{y/n})^n = [A^{x/n} + B^{y/n}][(A^{x/n})^{n-1} - (A^{x/n})^{n-2} B^{y/n} + (A^{x/n})^{n-3} (B^{y/n})^2 -...+ (A^{x/n})^2 (B^{y/n})^{n-3} - A^{x/n}(B^{y/n})^{n-2} + (B^{y/n})^{n-1}]$
For $A^x + B^y = C^z$ i.e.
$[A^{x/n} + B^{y/n}][(A^{x/n})^{n-1} - (A^{x/n})^{n-2} B^{y/n} + (A^{x/n})^{n-3} (B^{y/n})^2 -...+ (A^{x/n})^2 (B^{y/n})^{n-3} - A^{x/n}(B^{y/n})^{n-2} + (B^{y/n})^{n-1}] = C^z$
then every term in each of the 2 brackets must be integers and not irrational numbers. This therefore implies that $A^{x/n}$ and $B^{y/n}$ must be integers. Note that if A and B have a highest common factor of 1 i.e. they are coprime, then $A^{z/n}$ and $B^{y/n}$ have a highest common factor of 1 i.e. they are coprime. Let $A^{x/n} = P$ and $B^{y/n} = Q$. Hence we can re-write
$A^x + B^y = (A^{x/n})^n + (B^{y/n})^n = P^n + Q^n$
$= (P + Q)(P^{n-1} - P^{n-2} Q + P^{n-3} Q^2 -... + P^2 Q^{n-3} - PQ^{n-2} + Q^{n-1})$
Fermat's Last Theorem says that $P^n + Q^n \neq R^n$ i.e.
$(P + Q)(P^{n-1} - P^{n-2} Q + P^{n-3} Q^2 -... + P^2 Q^{n-3} - PQ^{n-2} + Q^{n-1}) \neq R^n$
Beal's conjecture of the form $A^x + B^y = C^z$ is therefore proved by using Fermat's Last Theorem. If P and Q on the left hand side of the inequality have a highest common factor $= k > 1$, this can be factored out to give
$(P + Q)(P^{n-1} - P^{n-2} Q + P^{n-3} Q^2 -... + P^2 Q^{n-3} - PQ^{n-2} + Q^{n-1})$
$= k^n(S + T)(S^{n-1} - S^{n-2} T + S^{n-3} T^2 -... + S^2 T^{n-3} - ST^{n-2} + S^{n-1})$
where $P= kS$ and $Q = kT$.
The introduction of the extra algebraic term, k^n is sufficient to allow
$k^n(S + T)(S^{n-1} - S^{n-2} T + S^{n-3} T^2 -... + S^2 T^{n-3} - ST^{n-2} + S^{n-1}) = A^x$
where $n \neq x$.

This proof of Beal's conjecture shows why some examples of

Beal's conjecture can always be manipulated to get new examples where 2 of the 3 terms have the same power/index. A few examples are

$27^4 + 162^3 = 9^7$ which can be re-written as $81^3 + 162^3 = 9^7$.

$7^6 + 7^7 = 98^3$ can be re-written as $7^3 + 7^4 = 14^3$

if $7^6 + 7^7 = 98^3$ is divided by 7^3.