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We wish to solve the equation

$$x^3 + px + q = 0$$

We seek a solution of the form $x = A + B$. then

$$A^3 + 3AB(A + B) + B^3 + p(A + B) + q = 0,$$

or

$$(A^3 + B^3 + q) + (3AB + p)(A + B) = 0.$$

We try to impose a solution

$$A^3 + B^3 = -q$$

$$AB = -\frac{p}{3}.$$

So the equation that we obtain is

$$A^3 - \frac{p^3}{27A^3} = -q.$$

or

$$(A^3)^2 + qA^3 - \frac{p^3}{27} = 0$$

Thus

$$A^3 = \frac{-q \pm \sqrt{q^2 + 4(p/3)^3}}{2} = -\frac{q}{2} \pm \sqrt{\left(\frac{q}{2}\right)^2 + \left(\frac{p}{3}\right)^3}$$

Which means

$$B^3 = -q - A^3 = -\frac{q}{2} \mp \sqrt{\left(\frac{q}{2}\right)^2 + \left(\frac{p}{3}\right)^3}.$$

We therefore conclude that we can choose

$$A^3 = -\frac{q}{2} + \sqrt{\left(\frac{q}{2}\right)^2 + \left(\frac{p}{3}\right)^3}$$

and the corresponding choice for B will be

$$B^3 = -\frac{q}{2} - \sqrt{\left(\frac{q}{2}\right)^2 + \left(\frac{p}{3}\right)^3}.$$

The solution x that we are looking for is therefore

$$x = \sqrt[3]{-\frac{q}{2} + \sqrt{\left(\frac{q}{2}\right)^2 + \left(\frac{p}{3}\right)^3}} + \sqrt[3]{-\frac{q}{2} - \sqrt{\left(\frac{q}{2}\right)^2 + \left(\frac{p}{3}\right)^3}}$$

We know that there must be 3 solutions to the cubic equation , and each number has 3 complex roots. At first glance this gives us 9 possibilities for x, But since $AB = -p/3$ we see that

$$arg(B) = arg(-p/3) - arg(A)$$

which assigns to each value of A a unique value of B.

If we write the equation in the form

$$x^3 + 3Px + 2Q = 0$$

and define

$$\Delta = \sqrt{Q^2 + P^3}$$

we obtain

$$A = \sqrt[3]{-Q + \sqrt{Q^2 + P^3}}$$

$$B = \sqrt[3]{-Q - \sqrt{Q^2 + P^3}}$$

where for each the three possible complex values of A the corresponding B is chosen so that $arg(B) = arg(-P) - arg(A)$.

$$x = \sqrt[3]{-Q + \sqrt{Q^2 + P^3}} + \sqrt[3]{-Q - \sqrt{Q^2 + P^3}}$$

the solution to the equation is then

$$x = \sqrt[3]{-Q + \Delta} + \sqrt[3]{-Q - \Delta}$$

Example: Find the roots of the equation :

$$x^3 - 3x - \sqrt{2} = 0.$$

Here $P = -1$ and $Q = -\frac{\sqrt{2}}{2}$. Then $Q^2 + P^3 = \frac{1}{2} - 1 = -\frac{1}{2}$. which implies

$$\pm\sqrt{Q^2 + P^3} = \pm\frac{\sqrt{2}}{2}i.$$

Therefore

$$A = \sqrt[3]{\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i} = \sqrt[3]{e^{i\frac{\pi}{4}}} \qquad \text{and} \qquad B = \sqrt[3]{\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}i} = \sqrt[3]{e^{-i\frac{\pi}{4}}}$$

Therefore

$$A = e^{i\frac{\pi}{12}}, e^{i\left(\frac{\pi}{12}+\frac{2\pi}{3}\right)}, e^{i\left(\frac{\pi}{12}+\frac{4\pi}{3}\right)} = e^{i\frac{\pi}{12}}, e^{i\frac{3\pi}{4}}, e^{i\frac{17\pi}{12}}.$$

$$B = e^{-i\frac{\pi}{12}}, e^{i\left(-\frac{\pi}{12}+\frac{2\pi}{3}\right)}, e^{i\left(-\frac{\pi}{12}+\frac{4\pi}{3}\right)} = e^{-i\frac{\pi}{12}}, e^{i\frac{7\pi}{4}}, e^{i\frac{15\pi}{12}}.$$

Which means that the roots of the polynomial are

$$x_1 = 2\cos\left(\frac{\pi}{12}\right) = \sqrt{2+\sqrt{3}},$$

$$x_2 = 2\cos\left(\frac{3\pi}{4}\right) = -\sqrt{2},$$

$$x_3 = 2\cos\left(\frac{17\pi}{12}\right) = -\sqrt{2-\sqrt{3}}.$$