The New Calculus for Dummies.

Definition: A **point** is the idea or concept of location or place. It is the **foundation** of all geometric objects.

Definition: A **geometric object** consists of one or more points.

Definition: A **path** describes a distance between any two points.

All the lines in the above diagram are visualizations of paths between the two points. There are an innumerable number (also known as **infinitely many**) of paths between any two points.

Definition: A path is said to be **continuous**, if no portion is disjoint from any other portion also on the path.

The previous diagrams shows paths that are discontinuous.
Definition: A *straight line* is a geometric object that describes the *shortest distance* or *path* between any *two* points.

**NOTE:** The *only* geometric object that possesses the attribute of *slope/gradient/inclination* is the straight line.

Definition: A straight line is said to be a *tangent line* to another geometric object (such as a curve) *if and only if*, a *portion* of the straight line intersects the curve in exactly *one point* and *does not cross the curve anywhere*.

![Tangent Line Diagram](image)

Definition: A path is said to be *smooth*, if and only if, *one* tangent line with *a defined gradient* can be constructed at *every point* in the path. A path is still considered smooth if there are points of inflection (points at which the concavity of the path/curve changes).

![Smooth and Non-Smooth Paths](image)

Example of a point of inflection:
Slope

Every straight line in a plane has two important characteristics:

- It describes the shortest path or distance between two points
- It has an angle of inclination that is measured from a horizontal line
Depending on the smaller angle of inclination, a straight line may have a slope or gradient. The slope is always determined by the smaller angle of inclination with the horizontal line. If both angles of inclination are right angles, the line (which is vertical) has no slope in terms of $\frac{\text{rise}}{\text{run}}$, which is Newton’s definition of slope. However, every line has a slope in terms of the Ancient Greek definition of slope, that is, a vertical line has a slope of $90^\circ$.

For any angle of inclination (except a right angle), it is possible to construct infinitely many right angled triangles by constructing a vertical line from the horizontal line that meets the straight line which is inclined to the horizontal line. See Fig. 2.

The slope of the straight line inclined to the horizontal line is determined by the ratio of the vertical (rise) and horizontal (run) sides of any of the given right angled triangles.
Thus, from Fig. 3, the slope is given by the ratio of the rise length over the run length:

\[ \text{slope} = \frac{\text{rise}}{\text{run}} \]

The slope ratio is constant for any of the right angled triangles formed by the given angle \( \theta \), since these triangles are similar. Therefore the slope can be determined by knowing the length of the rise and run or the angle, since

\[ \tan(\theta) = \frac{\text{rise}}{\text{run}}. \]

**Historical Note:**
The ancient Greeks calculated gradients by using right angles (90° angles or \( \frac{\pi}{2} \) radians). Although they also introduced tangents, they did not seem to be too interested in finding the slopes of these tangents or did not realize the significance thereof.
The tangent gradient or slope problem:

The process of finding the gradient of the tangent line is called differentiation. Before you continue reading, ponder this problem for a while.

The problem is easily solved if we can find the gradient of any secant on the path that is parallel to the blue tangent:

On first inspection, this is an almost impossible task. It took 330 years to find a rigorous method. In the New Calculus this is possible because of the secant theorem.

Before we study the secant theorem, let's take a look at some basic theory of gradients.

1. The only geometric object that has a gradient is the straight line. When we talk about the gradient (or the derivative) of a function at a
point, we mean the gradient of the tangent line at that point, provided it exists (can be constructed).

2. A gradient or slope is a comparison of the difference in the vertical distance (rise) between two points and the difference in the horizontal distance between the same two points with reference to a point of origin in a plane.

3. If $f$ is a planar function and $\left(c-m, f(c-m)\right)$ and $\left(c+n, f(c+n)\right)$ are the endpoints of a secant line intersecting $f$, then the gradient $k$ of the secant line is given by

$$k = \frac{f(c+n) - f(c-m)}{m+n} \quad \text{[G]}$$

where $c-m < c < c+n$. 
\( k \) is the ratio \( \frac{\text{rise}}{\text{run}} = \frac{k(m+n)}{m+n} \). In the new calculus, these magnitudes (rise and run) are well defined as differentials. If the rise is measured in terms of \( y \) and the run in terms of \( x \), then the rise and run differentials are \( dy \) and \( dx \) respectively, where

\[
dy = f(c+n) - f(c-m) \quad \text{and} \quad dx = m+n
\]

4. From \([G]\),

\[
k(m+n) = f(c+n) - f(c-m)
\]

Therefore,

\[
(m+n)(f(c+n) - f(c-m))
\]

Read as:

\( (m+n) \) divides the ordinate difference \( f(c+n) - f(c-m) \).

It follows that the magnitude \( k \) cannot contain \( (m+n) \) as a factor. Therefore the reduced form of the quotient \( \frac{f(c+n) - f(c-m)}{m+n} \) always contains exactly one term, that is, \( k \) which has no factor \( (m+n) \). The sum of the remaining terms is zero. As an example, suppose that the gradient \( k = 7 \), is given by:

\[
k = \frac{7(5) + 3(2+3)^2 + 2(2+3)^2 - (2+3)^3}{2 + 3}
\]

If \( m = 2 \) and \( n = 3 \), then

\[
k = \frac{7(m+n) + 3(m+n)^2 + 2(m+n)^2 - (m+n)^3}{m+n}
\]

\[
k = 7 + 3(5) + 2(5) - (5)^2 = 7
\]
The only term not containing \( m \) or \( n \) is 7. Since we know that \( k = 7 \), the sum of all the remaining terms must be zero:

\[
3(m+n) + 2(m+n) - (m+n)^2 = 0
\]

Check:

\[
3(5) + 2(5) - (5)^2 = 15 + 10 - 25 = 0
\]

Given the quotient \( \frac{f(c+n) - f(c-m)}{m+n} \), it follows that the gradient is given by those terms neither containing \( m \) nor \( n \).

The gradient does not depend on the values of \( m \) or \( n \).

If \( (c, l(c)) \) is a point on a straight line \( l \), its gradient is given by

\[
k = \frac{l(c+n) - l(c-m)}{m+n}
\]

but \( k \) is not dependent on either \( m \) or \( n \).

Proof:

The equation of \( l(x) \) is given by:

\[
l(x) = \frac{l(c+n) - l(c-m)}{m+n} x + F
\]

Substituting \( (c, l(c)) \) to find \( F \):

\[
l(c) = \frac{l(c+n) - l(c-m)}{m+n} (c) + F
\]

\[
F = l(c) - \frac{l(c+n) - l(c-m)}{m+n} (c)
\]

Therefore,
Since the left hand side of the last equation contains no terms in \( m \) or \( n \), the proof is complete.

The Secant theorem.

For any function \( f \) with a tangent line (at \( x = c \)) having gradient \( k \), the difference in a pair of secant line ordinates is always \( k(m + n) \) because of the gradient ratio. This implies the ordinate difference is always divisible by \( (m + n) \). Provided \( f \) is continuous and smooth over any interval \((c - m, c + n)\), there are infinitely many parallel secant line ordinate pairs \( f(c + n) \) and \( f(c - m) \), such that any secant line gradient \( \frac{f(c + n) - f(c - m)}{m + n} \) produces \( k \).

From the secant theorem, we define the derivative of \( f \) at \( x = c \) formally as \( f'(c) \): 

\[
l(x) = \frac{l(c + n) - l(c - m)}{m + n} (x) + l(c) - \frac{l(c + n) - l(c - m)}{m + n} (c)
\]

\[
\rightarrow l(x) = \frac{l(c + n) - l(c - m)}{m + n} (x - c) + l(c)
\]

\[
\frac{l(x) - l(c)}{x - c} = \frac{l(c + n) - l(c - m)}{m + n}
\]
\[
f'(c) = \frac{f(c + n) - f(c - m)}{m + n} = k + Q(m, n) \quad [D]
\]

where \( Q(m, n) \) is the sum of all the remaining terms.

**Examples:**

1. Find the gradient of the straight line given by \( l(x) = px + r \).

   \[
l'(c) = \frac{l(c + n) - l(c - m)}{m + n}
   \]

   \[
l'(c) = \frac{p(c + n) + r - p(c - m) - r}{m + n}
   \]

   \[
l'(c) = \frac{pm + pn}{m + n} = \frac{p(m + n)}{m + n} = p
   \]

   Once again we see that the slope (or gradient) of a straight line does not depend on the values of \( m \) or \( n \). A straight line is the only geometric object that has a slope or gradient. A tangent line cannot be tangent to any other straight line.

2. Find the gradient of the tangent line to the curve \( f(x) = ax^2 + bx + k \) at the point \( x = c \).

   \[
f'(c) = \frac{a(c + n)^2 + b(c + n) + k - (a(c - m)^2 + b(c - m) + k)}{m + n}
   \]

   \[
f'(c) = \frac{ac^2 + 2acn + bn + k - ac^2 + 2acm - am^2 - bc + bm - k}{m + n}
   \]

   \[
f'(c) = \frac{2acn + 2acm - am^2 + bn + bm}{m + n}
   \]

   \[
f'(c) = \frac{2ac(m + n) + a(n - m)(m + n) + b(m + n)}{m + n}
   \]
Since \( Q(0,0) = 0 \), the simplified gradient is \( f'(c) = 2ac + b \).

Note that question (2) could have been phrased as: Find the derivative of \( f(x) = ax^2 + bx + k \). It could also have been phrased as Find the gradient \( f(x) = ax^2 + bx + k \) at \( x = c \). However, it always means the gradient of the tangent line at the point \( x = c \). The function \( f(x) \) is a parabola and we know the only geometric object that has a gradient is the straight line.

\( f'(c) = 2ac + b \) is the general form of the derivative for the parabola. By replacing \( c \) with the \( x \)-coordinate of any other point, we arrive at the derivative at that point. A numeric (or value) derivative has a fixed value. For example, \( f'(2) = 2a(2) + b = 4a + b \). The value \( 4a + b \) never changes as opposed to \( 2ac + b \) which always depends on the value of \( c \).

3. If \( f'(c) = 2ac + b \), find expressions for \( dy \) and \( dx \).

We know that \( f'(c) = \frac{dy}{dx} = 2ac + b = \frac{2ac + b}{1} \).

Therefore \( dy = 2ac + b \) and \( dx = 1 \).

**Finding some simple derivatives:**

1. If \( f(x) = k \), where \( k \) is some constant, then

\[
f'(x) = \frac{k - k}{m + n} = 0
\]

2. \( f(x) = kx \)

\[
f'(x) = \frac{k(c + n) - k(c - m)}{m + n} = \frac{kn + km}{m + n} = k
\]

3. \( f(x) = kx^2 \)
\[ f'(x) = \frac{k(x+n)^2 - k(x-m)^2}{m+n} \]
\[ = \frac{kx^2 + 2kxn + kn^2 - kx^2 + 2kxm - km^2}{m+n} \]
\[ = \frac{2kx(m+n) + k(n^2 - m^2)}{m+n} \]
\[ = \frac{2kx(m+n) + k(n-m)(m+n)}{m+n} \]
\[ = 2kx + k(n-m) \]

but \( Q(m,n) = 0 \), so

\[ f'(x) = 2kx \]

4. \( f(x) = kx^3 \Rightarrow f'(x) = 3kx^2 \)

5. In general if

\[ f(x) = kx^n, \text{ then } f'(x) = knx^{n-1} \quad \text{(Power Rule)} \]

6.

\[ f(x) = \sqrt{x} \]
\[ \Rightarrow f'(x) = \frac{\sqrt{x+n} - \sqrt{x-m}}{m+n} \]
\[ \Rightarrow f'(x) = \frac{\sqrt{x+n} - \sqrt{x-m}}{m+n} \times \frac{\sqrt{x+n} + \sqrt{x-m}}{\sqrt{x+n} + \sqrt{x-m}} \]
\[ \Rightarrow f'(x) = \frac{x+n-(x-m)}{(m+n)(\sqrt{x+n} + \sqrt{x-m})} \]
\[ \Rightarrow f'(x) = \frac{m+n}{(m+n)(\sqrt{x+n} + \sqrt{x-m})} \]
\[ \Rightarrow f'(x) = \frac{1}{\sqrt{x+n} + \sqrt{x-m}} \]

but \( Q(m,n) = 0 \), so
\[ f'(x) = \frac{1}{\sqrt{x} + \sqrt{x}} = \frac{1}{2\sqrt{x}} \]

You could have gotten the same answer by using the result in no (5):

\[ f(x) = x^\frac{1}{2} \quad \therefore f'(x) = kx^{\frac{n-1}{2}} = 1 \times \frac{1}{2} \times x^{\frac{1}{2}-1} = \frac{1}{2} x^{-\frac{1}{2}} = \frac{1}{2\sqrt{x}} \]

7. \[ f(x) = \frac{1}{x} \]

\[ \Rightarrow f'(x) = \frac{1}{x+n} - \frac{1}{x-m} \]
\[ \Rightarrow f'(x) = \frac{x-m-(x+n)}{m+n(x-m)} \]
\[ \Rightarrow f'(x) = \frac{x-m-(x+n)}{(m+n)(x-m)} \]
\[ \Rightarrow f'(x) = \frac{-(m+n)}{(m+n)(x-n)(x-m)} \]
\[ \Rightarrow f'(x) = \frac{-1}{(x+n)(x-m)} \]

but \( Q(m,n) = 0 \), so

\[ f'(x) = \frac{-1}{x^2} \]

You could have gotten the same answer by using the result in no (5):

\[ f(x) = \frac{1}{x} \quad \therefore f'(x) = kx^{n-1} = 1 \times (-1) \times x^{-1-1} = -x^{-2} = \frac{-1}{x^2} \]

Exercises:

Find \( f'(x) \) in each one of the following exercises:
1. \( f(x) = \frac{1}{x^3} \)  
2. \( f(x) = \frac{1}{x^2} + x^3 + 9 \)  
3. \( f(x) = 0 \)

4. \( f(x) = (x + 4)^2 \)  
5. \( f(x) = \frac{3x^2}{4} \)  
6. \( f(x) = 5\sqrt{x(x-1)^2} \)

7. \( f(x) = (3x^2 + 4)^3 \)

**Differentiation Rules**

We know that \( f''(c) = \frac{dy}{dx} = \frac{f(c + n) - f(c - m)}{m + n} \).

Prove the **sum rule** using the New Calculus where \( f(x) = u(x) + v(x) \).

\[
f'(c) = \frac{u(c + n) + v(c + n) - [u(c - m) + v(c - m)]}{m + n}
\]

\[
f'(c) = \frac{u(c + n) - u(c - m) + v(c + n) - v(c - m)}{m + n}
\]

\[
f'(c) = \frac{u(c + n) - u(c - m)}{m + n} + \frac{v(c + n) - v(c - m)}{m + n}
\]

\[
f'(c) = u'(c) + v'(c)
\]

The proof of the **difference rule** is similar to the sum rule.

Prove the **product rule** using the New Calculus where \( f(x) = u(x)v(x) \).

\[
f'(c) = \frac{u(c + n)v(c + n) - u(c - m)v(c - m)}{m + n}
\]

Add 0 \( [u(c - m)v(c + n) - u(c - m)v(c + n)] \) to the numerator:
\[ f'(c) = \frac{u(c+n)v(c+n) - u(c-m)v(c+n) + u(c-m)v(c+n) - u(c-m)v(c-m)}{m+n} \]

\[ f'(c) = \frac{v(c+n)[u(c+n) - u(c-m)] + u(c-m)[v(c+n) - v(c-m)]}{m+n} \]

\[ f'(c) = \frac{v(c+n)[u(c+n) - u(c-m)]}{m+n} + \frac{u(c-m)[v(c+n) - v(c-m)]}{m+n} \]

\[ f'(c) = v(c+0)u'(c) + u(c-0)v'(c) = v(c)u'(c) + u(c)v'(c) \]

Prove the quotient rule using the New Calculus where \( f(x) = \frac{u(x)}{v(x)} \).

\[ f'(c) = \frac{\frac{u(c+n)}{v(c+n)} - \frac{u(c-m)}{v(c-m)}}{m+n} \]

\[ f'(c) = \frac{1}{m+n} \left[ \frac{u(c+n)}{v(c+n)} - \frac{u(c-m)}{v(c-m)} \right] = \frac{1}{m+n} \left[ \frac{u(c+n)v(c-m) - u(c-m)v(c+n)}{v(c+n)v(c-m)} \right] \]

Add 0 \( [u(c-m)v(c-m) - u(c-m)v(c-m)] \) to the numerator:

\[ f'(c) = \frac{1}{m+n} \left[ \frac{\{u(c+n)v(c-m) - u(c-m)v(c-m)\} - \{u(c-m)v(c+n) - u(c-m)v(c-m)\}}{v(c+n)v(c-m)} \right] \]

\[ f'(c) = \frac{1}{v(c+n)v(c-m)} \left[ \frac{\{u(c+n)v(c-m) - u(c-m)v(c-m)\} - \{u(c-m)v(c+n) - u(c-m)v(c-m)\}}{m+n} \right] \]
The chain rule is proved as follows:

Now, the chain rule in standard calculus states:

\[ f'(c) = g'[h(c)] \cdot h'(c) \quad \text{where} \quad f(c) = g[h(c)] \]

In the New Calculus, the chain rule can be proved as follows:

\[ f'(c) = \frac{g[h(c+n)] - g[h(c-m)]}{m+n} \]

\[ f'(c) = \frac{g[h(c+n)] - g[h(c-m)]}{(m+n)[h(c+n) - h(c-m)]} \cdot \frac{[h(c+n) - h(c-m)]}{1} \]

\[ f'(c) = \frac{g[h(c+n)]}{h(c+n) - h(c-m)} \cdot \frac{h'(c)}{h(c+n) - h(c-m)} - \frac{g[h(c-m)]}{h(c+n) - h(c-m)} \cdot \frac{h'(c)}{h(c+n) - h(c-m)} \]

\[ f'(c) = h'(c) \cdot \frac{g[h(c+n)] - g[h(c-m)]}{h(c+n) - h(c-m)} \]

Let \( p - q = h(c + n) - h(c - m) \).

We know from the New Calculus that in general :
For a good glimpse into academic ignorance and stupidity, read the following link which is a dissertation on the Chain rule! It is by James Franklin Cottrill: (http://homepages.ohiodominican.edu/~cottrilj/thesis.pdf)

Now for a proof of a special case of the chain rule using the New Calculus where $f(x) = [u(x)]^p$.

$$f'(c) = \frac{[u(c + n)]^p - [u(c - m)]^p}{m + n}$$

$$f'(c) = \frac{[u(c + n)]^{p-1}u(c + n) - [u(c - m)]^{p-1}u(c - m)}{m + n}$$

Add $0 \quad [u(c + n)]^{p-1}u(c - m) - [u(c + n)]^{p-1}u(c - m)$ to the numerator:

$$f'(c) = [u(c + n)]^{p-1}u'(c) + u(c - m)\frac{[u(c + n)]^{p-1} - [u(c - m)]^{p-1}}{m + n}$$
Continuing this way until the \( \frac{[u(c + n)]^{p-k} - [u(c - m)]^{p-k}}{m + n} = u'(c) \) where \( k = p - 1 \).

We have,

\[
f^*(c) = [u(c + n)]^{p-1}u'(c) + [u(c + n)]^{p-2}u'(c) + u(c - m)[u(c + n)]^{p-3}u'(c) + \ldots + u(c - m)^{p-1}u'(c)
\]

\[
f^*(c) = u'(c)[u(c + n)]^{p-1} + u(c - m)[u(c + n)]^{p-2} + u(c - m)^2[u(c + n)]^{p-3} + \ldots + u(c - m)^{p-1}
\]

\[
f^*(c) = u'(c)[u(c + 0)]^{p-1} + u(c - 0)[u(c + 0)]^{p-2} + u(c - 0)^2[u(c + 0)]^{p-3} + \ldots + u(c - 0)^{p-1}
\]

\[
f^*(c) = u'(c)[u(c)]^{p-1} + [u(c)]^{p-2} + [u(c)]^{p-3} + \ldots + [u(c)]^{p-1}
\]

\[
f^*(c) = p[u(c)]^{p-1}u'(c)
\]